FLUID MECHANICS

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BY

RUSSELL A. DODGE

Professor of Engineering Mechanics,
University of Michigan

AND

MILTON J. THOMPSON

Professor and Chairman, Department of Aeronautical Engineering,
and Associate Director—Defense Research Laboratory,
University of Texas

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PREFACE

Since the beginning of the twentieth century there has been a great increase of knowledge pertaining to the mechanics of fluids, and during this period some of the progress in all branches of engineering has been founded on or assisted by a better understanding of fluid motion. This fact, coupled with the evident interest of engineering students in fluid flow, suggests that some of this new knowledge should, if possible, be made available to undergraduates. Much of the work in fluid mechanics has been of great mathematical complexity but it is believed that the material presented here is in such a form that it can be mastered by the average student having the usual foundation in physics, mathematics and mechanics.

Since the fundamental principles governing flow are the same for both gaseous and liquid fluids it is logical to begin the study of flow of both types of fluids in one course. Furthermore such a treatment is economical of teaching time. In order to emphasize the universal character of fluid mechanics the problems discussed in this book have been treated, wherever practicable, from a rational point of view. Purely empirical methods are mentioned but briefly and in general are considered as a means for supplying necessary coefficients and exponents appearing in the equations developed. Tables of coefficients have been limited to the number necessary to illustrate discussions and solve representative problems. Rational methods have been emphasized by the frequent use of the principles of dimensional analysis in developing theory. In the study of fluid motion, stress has been laid on the significance of the dimensionless parameters, Reynolds' number for viscous fluids, Mach's number for compressible fluids, and Froude's number for flow with a free surface.

The subject matter in the book naturally divides itself into statics and dynamics of fluids. The latter division contains discussions of flow relative to the usual external boundaries, such as pipes and channels, but it differs from older treatments in that it also deals with flow relative to internal boundaries, that is, flow around objects such as airfoils and ship hulls. Most of the usual elementary hydraulics has been retained but it is

presented with emphasis on the fluid mechanics involved. The book undoubtedly contains more material than can be covered in a single course, but the arrangement is such that the instructor can select those subjects which best fit the needs and interests of his students. In the case of the more advanced subjects the discussion here must be regarded as only an introduction to these extensive fields. For example, the treatment of the thermodynamics of compressible fluid flow is sufficient to show the application of fluid mechanics but is not intended to be complete.

In preparing this book the authors have drawn extensively upon technical literature. Credit is due to the authors, manufacturers and professional societies who have kindly granted permission to use their material. References to publications are given in the text and it is hoped that the acknowledgments are complete, but the National Advisory Committee for Aeronautics deserves special mention in this connection. The name of Professor L. Prandtl of the University of Göttingen is referred to more frequently than any other. Because of his pioneering work and continued valuable contributions in the field of fluid mechanics, this will seem entirely natural to those who are familiar with the development of the science. Probably no single work has been drawn upon more freely than "Applied Hydro- and Aero-Mechanics," by Professor Prandtl and Dr. O. G. Tietjens, and we gladly recognize our indebtedness to these authors.

It is also a pleasure to express our thanks for the valuable aid of our colleagues and associates in the College of Engineering of the University of Michigan. In particular we acknowledge the many suggestions made by Professors R. T. Liddicoat, H. M. Hansen and L. A. Baier. The encouragement given and the interest shown by Professor E. L. Eriksen have been very helpful. A substantial contribution was that of Professor R. P. Harrington of the Polytechnic Institute of Brooklyn, who read the proof and made many constructive criticisms. Finally we wish to express our appreciation of the valuable services of Mrs. Helen M. Anderson and Miss Reta E. Morden who typed the manuscript, and of Messrs. R. S. Frazier, P. E. Theobald and T. L. Vander Velde, who assisted in preparing the drawings.

Ann Arbor, Michigan,
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Russell A. Dodge,
Milton J. Thompson.

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FLUID MECHANICS

CHAPTER I

PROPERTIES OF FLUIDS

1. Nature of Fluid Substance.—The term fluid is applied to substances which, owing to the nature of their internal structure, offer comparatively little resistance to a change in form. Fluids are commonly classified as incompressible or compressible, that is, as liquids or gases. Liquids offer great resistance to change in volume while gases have little resistance to change in either form or volume.

A gas occupies all the space in which it is contained. Its volume is reduced by an increased pressure or compressive force, the pressure within the gas increasing until it just balances the applied force. No fluid is capable of any internal adjustment which will enable it to maintain equilibrium at rest while subjected to a shear stress, however small. If a shearing force is applied to any fluid, the fluid will continue to move as long as it is applied. There will invariably be some movement in which the velocity is proportional to the applied shear stress. The relationship between force and velocity depends, among other things, upon that property of fluids known as viscosity.

Liquids and gaseous fluids are quite different in compressibility and in the existence of the free surface in the former. The behavior of all flowing fluids, whether liquids or gases, is quite similar and, when the flow is in conduits, that is, flow without a free surface, the behavior is identical for most conditions.

An ideal or perfect fluid is merely one which, for purposes of developing theory or making a mathematical demonstration, is conveniently assumed to be nonviscous or incompressible or both. Such fluids do not exist and theory based upon such assumptions is subject in its application to correction for the effect of physical properties that have been neglected.

2. Density of Fluids.—The density of a substance is the mass of a unit of volume under certain specified temperature and pressure conditions. This is not to be confused with weight per unit volume or specific weight. If density is ρ and weight per unit volume is w, then

$$\rho = \frac{w}{g} \qquad \text{or} \qquad w = \rho g$$

g being the acceleration of gravity. If w is the specific weight in pounds per cubic foot and g the acceleration of gravity in feet per second per second, then

$$\rho = \frac{w}{g} \approx \frac{\text{lb.}}{\text{ft.}^3}\,\frac{\text{sec.}^2}{\text{ft.}} \qquad \frac{\text{lb.sec.}^2}{\text{ft.}^4} \approx \frac{M}{L^3}$$

in which the sign   is used to denote dimensional equality. Similarly,

$$w = \rho g \approx \frac{M}{L^2T^2}$$

The symbols M, L and T are used to represent the fundamental units of mass, length and time, respectively. The unit of mass in the English system is known as the slug, so that density is measured in slugs per cubic foot. The ratio of weight to mass, g, is the acceleration of gravity and in the English system is about 32.2 ft. per sec. per sec. Its value varies slightly from place to place on the earth's surface. The specific weight of fresh water at ordinary temperatures is about 62.4 lb. per cu. ft.

Specific volume is the volume per unit weight. It is the reciprocal of specific weight, its dimension being in cubic feet per pound or, in fundamental units, L^2T^2/M.

Problem 1. What are the dimensions of density and specific weight in the metric system, using centimeters, grams weight and seconds? What are the density and specific weight of fresh water in these units?

2. Compute the specific weight and density of a liquid which weighs 2000 lb. per cu. m. What is its specific volume in English units?

3. Compressibility of Fluids.—All fluids are compressible to some extent. Liquids are only slightly compressible, so little in fact that in most ordinary problems they are considered to be incompressible. In some problems, for example, those dealing with water hammer, the compressibility is an important factor.

The modulus of elasticity of water in compression is about 300,000 lb. per sq. in. at ordinary pressures. It increases slightly with temperature and at extremely high pressures becomes much larger.

Gaseous fluids are highly compressible and are therefore subject to a wide variation in density. The distinctive behavior of gases in this respect bears less resemblance to that of liquids than any other property of gases. The relationship of pressure, volume and temperature in gaseous fluids is discussed in the following pages.

4. Boyle's Law.—If a quantity of fluid is placed in a closed container, there will in general be a force acting on the walls and bottom of the vessel owing to the fact that the fluid is not capable of maintaining itself in a fixed shape but tends to flow as soon as the restraining walls are removed. The force produced by the fluid on the walls of the container is usually studied by noting the value of the pressure at various points, this latter quantity being the force acting on a unit area of surface. The action of the fluid on the walls of the container is, of course, accompanied by a reaction on the fluid which is manifested by the existence of pressures at all points within the fluid. In the case of a liquid the pressure at a point beneath the surface is due primarily to the weight of the fluid above it, while in the case of a gas the pressure is produced by a combination of this effect and the activity of the molecules. Thus a volume of gas placed in a closed tank may undergo an increase in pressure if the gas is heated from an external source.

The dimensions of a pressure are easily determined if it is recalled that this quantity is equivalent to a force acting on a unit area. Thus

$$p = \frac{F}{A} \approx \frac{\frac{ML}{T^2}}{L^2} \approx \frac{M}{LT^2}$$

In the usual English units, pressures are measured in pounds per square foot or per square inch.

The rule governing the pressure-volume relationship at constant temperature is known as Boyle's law. It is expressed by the equation

$$\frac{p_1}{p} = \frac{v}{v_1} \qquad \text{or} \qquad p_1v_1 = pv = C \tag{1}$$

in which p and p_1 are pressures on the absolute scale and v and v_1 are the corresponding volumes, while C is a constant.

Since the density of a given body varies inversely as its volume, the former value is directly proportional to the pressure, that is,

$$\frac{p}{\rho} = \frac{p_1}{\rho_1} \quad \text{or} \quad \frac{p}{w} = \frac{p_1}{w_1} \tag{2}$$

Expressed in words, Boyle's law states that the volume of a body of gas varies inversely as the pressure when the temperature is constant, that is, when conditions are isothermal.

Problem 3. A cylinder having a volume of 2.5 cu. ft. contains 34 lb. of gas at an absolute pressure of 250 lb. per sq. in. If the gas is compressed at constant temperature to one-half of its original volume, what will be its pressure? Compute the density of the gas before and after compression.

5. Gay-Lussac's or Charles' Law.—Gay-Lussac verified experimentally the theory first advanced by Charles that gases under constant pressure expand in proportion to an increase in temperature and that all gases have the same coefficient of expansion.

The volume of a given mass of gas at any temperature t is

$$v_t = v_0(1 + \alpha_p t) \tag{3}$$

in which v_0 is the volume of the given mass at zero temperature and α_p is the coefficient of expansion at constant pressure. If the volume of the same given mass of gas is kept constant, the pressure increases in proportion to the increase in temperature, the new pressure at temperature t being

$$p_t = p_0(1 + \alpha_v t) \tag{4}$$

in which p_0 is the pressure at zero temperature and α_v is the pressure coefficient at constant volume.

The relationship between the coefficients α_p and α_v may be determined by considering a mass of gas at the initial conditions represented by p_0, v_0 and $t_0 = 0$. This gas is to be changed to the conditions represented by p, v and t. This change might be effected by first holding the temperature constant and changing the pressure to the value p. If the corresponding volume is v', then according to Boyle's law

$$pv' = p_0 v_0 \tag{5}$$

The remaining step in this transformation may be effected by changing the temperature from zero to t at the constant pressure p so that, on applying Eq. (3) and noting that the volume change is from v' to v, the following expression is obtained:

$$v = v'(1 + \alpha_p t) \tag{6}$$

Now Eqs. (5) and (6) may both be solved for the value of the intermediate volume v', and on equating these values the result obtained is

$$pv = p_0 v_0(1 + \alpha_p t) \tag{7}$$

The same change could also be accomplished by first changing the temperature from zero to t at constant volume. If the corresponding pressure is p'', then

$$p'' = p_0(1 + \alpha_v t) \tag{8}$$

The pressure may now be increased from p'' to p at constant temperature, in which case

$$p'' v_0 = pv \tag{9}$$

When values of p'' obtained from Eqs. (8) and (9) are equated, the result is

$$pv = p_0 v_0(1 + \alpha_v t) \tag{10}$$

Since Eqs. (7) and (10) represent the same final conditions for the gas, it follows at once that the values of α_p and α_v must be equal, that is, $\alpha_v = \alpha_p = \alpha$.

According to Eq. (4), the temperature at which the pressure in the gas drops to zero is $t = -\frac{1}{\alpha}$ and this point is known as the absolute zero of the temperature scale. It represents a limiting condition at which all molecular activity in the gas has ceased. On the Fahrenheit scale its value is $-459.4°$, while on the centigrade scale it is $-273°$. It is frequently convenient to study the behavior of a gas using the temperature T measured from the absolute zero point rather than from some arbitrary datum as in the case of the usual temperature scales. The relation between the temperatures t and T is simply

$$T = t + \frac{1}{\alpha} \tag{11}$$

Problem 4. Compute the value of the coefficient of expansion, α, for the centigrade and Fahrenheit scales.

6. The Combined Gas Law.—The statements given in the preceding articles may be combined into a single equation which represents the behavior of a gas under all possible conditions. This equation is known as the combined gas law or the equation of state. If the coefficients α_p and α_v are replaced by the single term α, then Eqs. (7) and (10) become identical in form, that is,

$$pv = p_0v_0(1 + \alpha t)$$

or, if the absolute temperature from Eq. (11) is introduced,

$$pv = p_0v_0\alpha T.$$

If v and v_0 are specific volumes, then the quantity $p_0v_0\alpha$ is a constant for any given gas depending only on the conditions when t is zero and on the value of the coefficient of expansion. If this constant is designated by the single symbol R, then

$$pv = RT \tag{12}$$

The numerical value and the dimensions of R depend upon the gas, the temperature scale and the pressure and volume adopted as a standard. With these fixed, R is a constant. If v_0 is the specific volume in cubic feet per pound and p_0 is atmospheric pressure in pounds per square foot, then p and v must have corresponding dimensions so that

$$R = \frac{pv}{T} \approx \frac{\text{lb.}}{\text{ft.}^2}\,\frac{\text{ft.}^3}{\text{lb.}}\,\frac{1}{\text{degree}} \approx \frac{\text{ft.}}{\text{degree}}$$

The value of R for dry air is 96.1 ft. per °C. or 53.3 ft. per °F.

Equation (12) is known as the equation of state for gases. Dividing both members by the corresponding members of the same equation for any other temperature T_1,

$$\frac{pv}{p_1v_1} = \frac{T}{T_1} \tag{13}$$

It follows from Eq. (12) or (13) that for constant pressures the volume of a body of gas is proportional to the absolute temperature. This is another way of stating Charles' law which, expressed algebraically, is

$$Tv_1 = T_1 v \tag{14}$$

It also follows from these equations that for constant volumes the pressure is proportional to the absolute temperature. If T equals T_1, Eq. (13) becomes Eq. (1), $p_1 v_1 = pv$, or Boyle's law.

Problem 5. A mass of air having a volume of 3.75 cu. ft. at a temperature of 519°F. abs. is allowed to expand until the pressure is 20 per cent of the original value and its temperature is 480°F. abs. What is its volume after expansion?

6. What is R for air in Eq. (12) when dealing with pressure in pounds per square inch, specific volume in cubic feet per pound and temperature in degrees centigrade?

7. Hydrogen has a specific gravity of 0.0000899 at 14.7 lb. per sq. in. pressure and at 59°F. What is the value of R?

8. Air has a specific weight of 0.0765 lb. per cu. ft. at 59°F. and 14.7 lb. per sq. in. What is its specific weight at an altitude of 10,000 ft. where the pressure is 10.1 lb. per sq. in. and the temperature is 21.2°F.?

7. Limitations of Gas Laws.—The laws developed in the foregoing articles are correct for a "perfect gas." For real gases, they are accurate at ordinary temperatures and for relatively small changes in pressure or volume. It is possible to liquefy any gas by the application of a large pressure with the gas at low temperature. It is obvious that no gas obeys the gas laws at the point of liquefaction. As the gas approaches liquefaction there is a wide variation in its behavior from the laws. Thus the behavior of gases which liquefy readily at ordinary temperatures diverges widely from the laws while gases like nitrogen, oxygen and hydrogen follow the laws closely at usual temperatures and for small pressure changes. Although air is a mixture of gases, it follows the gas laws very closely.

8. Adiabatic Expansion or Compression.—In Art. 4 it was noted that Boyle's law applies only to an isothermal change in volume, that is, a change in volume without change in temperature. If the expansion or compression is accomplished in such a way that there is no transfer of heat from or to the gas, the conditions are said to be adiabatic. Under such conditions there is a change in temperature and the pressure-volume relation follows the equation

$$p_1 v_1^k = pv^k = \text{a constant} \tag{15}$$

where $k = c_p/c_v$, c_p being the specific heat of the gas at constant pressure and c_v the specific heat at constant volume. The ratio

c_p/c_v is also the ratio of the thermal capacities of the gas under these constant conditions. The value of k for dry air, hydrogen and oxygen is 1.406. For additional values of k see Table I.

TABLE I.—VALUES OF k FOR VARIOUS GASES

Acetylene	1.28	Chlorine	1.32
Ammonia	1.32	Methane	1.32
Carbon dioxide	1.31	Nitrogen	1.41
Carbon disulphide	1.21	Steam at 100°C.	1.33

The ratio $k = c_p/c_v$ is also used as a factor in determining the velocity of sound in a gas, a quantity which is sometimes an important consideration in dealing with the flow of fluids.

Problem 9. A gas having a specific volume of 12.3 cu. ft. per lb. at initial conditions of 20 lb. per sq. in. pressure is compressed until the pressure is 100 lb. per sq. in. Draw curves showing the relation between pressure and specific volume (*a*) for isothermal compression, (*b*) for adiabatic compression with $k = 1.4$.

9. Vapor Pressure.—At the free surface of a liquid there is a constant motion of molecules away from the surface. This process of vaporization from a liquid is evaporation. If the space above the free surface is enclosed it will become saturated with vapor and there will be no further increase of the number of molecules in the vapor state, but the interchange of molecules between liquid and vapor at the free surface continues.

In the saturated state the pressure of the vapor above the liquid is equal to the vapor tension at the liquid surface and a state of equilibrium exists. Any enclosed space will become

TABLE II.—VAPOR PRESSURE OF WATER AT VARIOUS TEMPERATURES

Temperature, °C.	p_v/w, feet of water	Temperature, °C.	p_v/w, feet of water
−20	0.042	50	4.100
−10	0.096	60	6.630
0	0.203	70	10.39
10	0.407	80	15.38
20	0.773	90	23.40
30	1.403	100	33.91
40	2.450		

saturated if the quantity of liquid is sufficient. Vapor tension, and therefore vapor pressure, increases with temperature and at the boiling point is equal to atmospheric pressure. If the enclosed space above a free surface is filled, at pressure p_a, with a gas that does not react with the liquid, then vapor will escape until the pressure is increased by p_v and the new pressure in the space is $p = p_a + p_v$. This fact follows from Dalton's law. Values of vapor pressure of water are given in Table II.

10. Viscosity.—Viscosity is often defined as stickiness or treacliness of a substance. A descriptive definition of viscosity, however, is not sufficient for the purpose of fluid mechanics, an adequate physical and mathematical conception of the property being necessary to the understanding of fluid flow. Viscosity is the property of a fluid by virtue of which it offers resistance to shear stress and is in linear proportion to the ability of the fluid to resist such stress.

Since all fluids, both liquid and gaseous, have viscosity, the property must be explained by one or more physical properties common to all fluids, such as molecular activity. Owing to molecular activity there is a constant interchange of molecules and therefore of momentum between contiguous layers of fluid. It can be shown that, if the adjacent layers are moving with different velocities, this constant interchange of momentum sets up a resistance to any relative motion of the two layers. Energy is transformed to heat and a steady force is required to replace this energy and maintain the velocity.

The viscosity of gases increases with temperature. When the temperature of a gas is raised, the molecular activity and the rate of interchange of molecules between layers are increased with a resulting increase in force necessary to maintain the relative motion of adjacent layers.

The viscosity of a liquid decreases with temperature. The interchange of molecules is accelerated by an increase in temperature just as in gases but the viscosity of a liquid must be regarded as the combined effect of cohesion and interchange of momentum. A change in temperature has opposite effects on cohesion and molecular activity, the effect on cohesion being more pronounced, so that the greater cohesion at low temperatures results in a greater viscosity. Thus temperature changes have opposite effects upon the viscosity of liquids and gases.

When a shearing force or stress is applied to elastic material there is a definite deformation which is proportional to the force. A shearing force applied to viscous material causes continuous and unlimited deformation but at a rate proportional to the force. The rate of shear under known conditions then becomes a measure of the viscosity of a fluid and is used indirectly in the technique of determining viscosity experimentally, that is, in viscometry. The behavior of a substance under shearing stress is the only criterion by which it may be classified as fluid or solid. Any material in which a continuous deformation is caused by a shearing force, however small the force, must be a fluid. From this statement, the conclusion may be drawn that any fluid in equilibrium is in a state of complete freedom from shear stress.

Plastic materials are sometimes erroneously considered to be merely very viscous substances. Unlike viscous fluids, plastic materials are not moved by indefinitely small shear forces. When they do move, the distribution of velocity is quite different from that in viscous materials. This difference is due to the fact that the motion of plastic materials is influenced by ordinary friction of solids, a factor that does not exist in fluid motion.[1]

An idea of the importance of viscosity in certain types of flow may be gained from the facts that water at 100°F. is about thirty-six times as viscous as air, a light motor oil is over sixty times as viscous as water and the viscosity of water at the freezing point is six times that at the boiling point.

A mathematical discussion of viscosity will be found in Chap. VIII in which numerical values are given.

[1] For a thorough discussion of viscosity and plasticity, see E. C. Bingham, "Fluidity and Plasticity," McGraw-Hill Book Company, Inc., New York, 1922, or Emil Hatschek, "The Viscosity of Liquids," G. Bell & Sons, Ltd., London, 1928.

CHAPTER II

STATICS OF FLUIDS

11. Static Fluid.—A body of fluid at rest or moving bodily with uniform velocity is in equilibrium and may be said to be a static fluid. In this condition it is not changing its shape in any way and every part of it must therefore be free of any stress that might cause motion of or deformation of particles within the body. There are three kinds of stress to which any body may be subjected, namely, compression, tension and shear.

It is the nature of fluids that they move continuously under shear stress and cannot withstand tensile stress. It is well

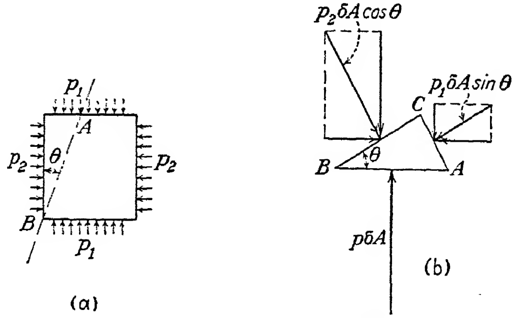


FIG. 1.—Free bodies of fluid.

known that fluids are capable of withstanding compressive stress, which is usually called pressure when associated with fluids. This compressive stress, or pressure, must be applied to fluids in a manner that is special as compared to stresses in solids. Compressive stresses will now be applied to a free body of fluid with the purpose of finding this special nature.

Figure 1*a* represents an infinitesimal particle of fluid which for convenience is taken to be in the form of a cube. Its horizontal faces are subjected to a compressive stress p_1 and the vertical sides to a compressive stress p_2. Figure 1*b* shows a new free body cut off from this particle by a plane AB which makes any angle θ with the vertical. If the area of face AB is taken as δA, the forces on faces AC and CB are $p_1\ \delta A \sin\theta$ and $p_2\ \delta A \cos\theta$. Since the body is a particle of static fluid, that is,

fluid without shear stress, the force on AB must be a normal one, the total force being the pressure p times the area δA or $p\ \delta A$. Taking horizontal components of forces and applying the laws of equilibrium, it follows that

$$p_1\ \delta A \sin\theta \cos\theta = p_2\ \delta A \cos\theta \sin\theta$$

and

$$p_1 = p_2$$

Taking vertical components of forces

$$p\ \delta A = p_1\ \delta A \sin^2\theta + p_2\ \delta A \cos^2\theta$$
$$p = p_1(\sin^2\theta + \cos^2\theta)$$

and

$$p = p_1$$

so that

$$p = p_1 = p_2 \tag{1}$$

Equation (1), expressed in words, states that the same pressure is acting on the three faces of ABC. Noting that θ is any angle whatever and that the particle might be in any position, it is seen that the pressure at any point within the fluid is the same on every plane through the point. It may then be stated that the pressure at any point in a static fluid is the same in every direction. This important principle is known as Pascal's law.

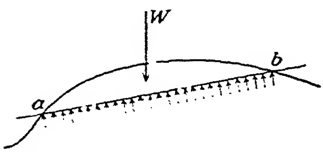


FIG. 2.—Free surface not in equilibrium.

The pressure exerted on the wall of a container by a static fluid is equal and opposite to that exerted on the fluid by the container. It has been shown that the latter must be normal to the fluid surface, hence the pressure of a static fluid on a surface must be normal to the surface at every point.

The free surface of a static liquid is always horizontal. The truth of this statement can be demonstrated by assuming a surface in any other position, in which case it would be possible to cut away a free body by some inclined plane such as ab in Fig. 2. Statically the only forces acting on the body of liquid above the plane are the force of gravity in the vertical direction and the pressure normal to the inclined plane. These forces cannot satisfy the laws of statics and hence such a free body cannot exist in a static liquid. The surface must therefore be

horizontal. A non-horizontal surface may appear to exist in a body of very viscous and apparently static fluid, but the existence of such a surface is evidence that conditions are not static and that flow is taking place, however slowly.

12. Relation of Pressure to Elevation.—The effect of elevation on pressure in a static fluid can be determined by considering the forces on a free body of the fluid. Figure 3 is such a body in the form of a vertical right cylinder of cross-sectional area A and height h, located anywhere within a larger body of static fluid. The only vertical forces acting on the body are the pressure forces on the ends, p_1A and p_2A, and the weight whA. These forces must satisfy the laws of statics, whence

$$p_1A - p_2A = whA$$

and

$$p_1 - p_2 = wh \tag{2}$$

FIG. 3.—Cylindrical free body of fluid.

It appears from Eq. (2) that the difference in pressures at two points in a static liquid is in direct proportion to the difference in elevation, the pressure always increasing with the depth. This is strictly true for fluids of uniform density and approximately correct in gases when the pressure difference is so small that the density is nearly constant.

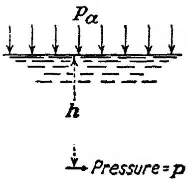


FIG. 4.—Pressure-depth relation.

It may be seen from Eq. (2) that the pressure in a body of static fluid is constant throughout a horizontal plane. It can also be shown that the density of a static fluid is constant in any horizontal plane.

The result expressed in Eq. (2) is independent of the relative horizontal position of points 1 and 2 and can be obtained by considering the equilibrium of a prism in any inclined position.

The pressure at the free surface of a liquid is that exerted on it by the gaseous fluid above. This is usually pressure due to weight of the atmosphere and is commonly called atmospheric pressure, p_a.

The pressure at any point in a liquid can be obtained by applying Eq. (2) to Fig. 4. Then the pressure at depth h is

$$p = p_a + wh \tag{3}$$

13. Pressure Units and Scales.—Pressures may be expressed in various units and on different scales, the choice of units and scales depending upon custom or convenience in the work at hand.

In the equation

$$p = p_a + wh$$

the pressure p is said to be on the absolute scale. If the pressure at the free surface is called zero, then

$$p = wh \tag{4}$$

and p is said to be on the gage scale. Gage pressure is the pressure relative to atmospheric pressure. The difference between a pressure expressed on the absolute scale and the same pressure on the gage scale is the constant p_a, having a value at sea level under normal conditions of 2116 lb. per sq. ft. or 14.7 lb. per sq. in. In using Eqs. (2), (3) and (4), units must be chosen so as to make them dimensionally homogeneous. Thus if A and h are in feet, w will be in pounds per cubic foot and p in pounds per square foot. When the liquid is fresh water, Eqs. (3) and (4) become

$$p \text{ (lb. per sq. ft. abs.)} = 2116 + 62.4h \tag{5a}$$
$$p \text{ (lb. per sq. ft. gage)} = 62.4h \tag{5b}$$

and, dividing by the number of square inches in 1 sq. ft.,

$$p \text{ (lb. per sq. in. abs.)} = 14.7 + \frac{62.4}{144}h = 14.7 + 0.433h \tag{6a}$$
$$p \text{ (lb. per sq. in. gage)} = 0.433h \tag{6b}$$

The atmosphere is a common unit of pressure which is convenient in dealing with compressible fluids. Normal atmospheric pressure at sea level at a temperature of 15°C. or 59°F. is taken as 1 atmosphere. It is equivalent to 2116 lb. per sq. ft. or 14.7 lb. per sq. in. on the absolute scale. A pressure of 3 atmospheres, for example, is $3 \times 2116 = 6348$ lb. per sq. ft. abs.

On the gage scale it is possible to have negative pressures, that is, pressures less than zero. The maximum possible negative pressure is $-p_a$. The term vacuum is commonly used to indicate negative pressures. A space in which the pressure is less than atmospheric is said to be under a vacuum even though

the space is entirely filled with gas or liquid. For example, a pressure of 5 lb. per sq. in. abs., is −9.7 lb. per sq. in. gage, or a vacuum of 9.7 lb. per sq. in.

Problem 10. Derive the relation of Eq. (2) between pressure and elevation by considering the equilibrium of an inclined prism of liquid of constant small cross section having both ends submerged.

14. Equivalent Head.—From Eq. (4) it is evident that the depth of liquid required to produce a given gage pressure is

$$h = \frac{p}{w} \tag{7}$$

The expression p/w is called the pressure head and it is common practice to indicate pressure or vacuum by stating the corresponding head. If p is in pounds per square foot and w is the weight per cubic foot of liquid, h becomes feet of head. Pressure head is also stated in inches of mercury, feet of water or any other convenient unit of length of fluid column.

Example.—The water in a salt sea weighs 68 lb. per cu. ft. What is the absolute pressure in pounds per square inch at a depth of 20 ft. when the atmospheric pressure is 2080 lb. per sq. ft.? At what depth in fresh water would the pressure be the same?

Solution.—The absolute pressure at any point in the water is the absolute pressure at the free surface plus the increase in pressure due to the depth of water. Then

$$2080 + 20(68) = 3440 \text{ lb./sq. ft. abs.}$$
$$3440 \div 144 = 23.89 \text{ lb./sq. in. abs.}$$

The pressure at depth h in fresh water is $62.4h$. Then $62.4h = 20(68)$ and $h = 21.79$ ft.

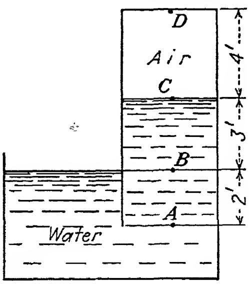


FIG. 5.

Problem 11. What is the pressure in pounds per square foot absolute and gage at a point 10 ft. below a free water surface?

12. What is the pressure in pounds per square foot absolute and gage at a point 10 ft. below the free surface of a liquid having a specific gravity of 3?

13. At sea level the atmosphere has specific weight of 0.0765 lb. per cu. ft. What is the absolute pressure in pounds per square inch at elevations of 10 ft. and 1000 ft. above sea level, assuming the atmosphere to be of constant density?

14. Compute the pressure in pounds per square inch gage at points A, B, C, D, in Fig. 5.

15. At what distances below a free water surface are the pressures 1 atm. gage and 3 atm. absolute?

16. Find the equivalent of a pressure of 10 lb. per sq. in. gage (*a*) in feet of water, (*b*) in inches of mercury.

17. Find the equivalent of a vacuum of 4 lb. per sq. in. (*a*) in inches of water, (*b*) in pounds per square inch absolute.

15. Standard Air.—In the preceding articles equations have been developed which express relations between pressure, volume, temperature and density of gases. In order to get numerical results from these equations it is necessary to start from known values of these properties under certain standard conditions, and in scientific work it is common to consider the properties at 0°C. and at a pressure of 1 atmosphere as standard.

The advance of aeronautics and the necessity for comparing performances of aircraft under standard conditions have led to the adoption of a so-called "standard air." Standard air is air at 59°F. or 15°C. under a pressure of 29.92 in. or 760 mm. of mercury (2116 lb. per sq. ft.). Under these conditions air has the following properties:

Specific weight $w = 0.07651$ lb. per cu. ft.
Specific volume $v = 1/w = 13.07$ cu. ft. per lb.
Density $\rho = 0.002378$ slug per cu. ft.
Viscosity $\mu = 3.723 \times 10^{-7}$ lb. sec. per sq. ft.

The temperature of the atmosphere reduces with altitude at an average rate of 3.57°F. per thousand feet. Thus the temperature at any elevation, z ft., is

$$t = 59 - 0.00357z \text{ (°F.)} \tag{8a}$$

or

$$T = 518.4 - 0.00357z \text{ (°F. abs.)} \tag{8b}$$

The temperature gradient is practically constant to an altitude of about 35,300 ft. At this altitude the temperature is −67°F. and so far as is known remains constant at greater altitudes. The actual temperature, and therefore density and pressure, may vary considerably from those of standard air at sea level. The lower atmosphere in which the temperature varies is known as the troposphere, and the upper stratum at constant temperature as the stratosphere. There is no sharp division between the two, the change from one to the other being in the nature of a gradual transition. While the change to isothermal conditions takes place at an altitude of 35,300 ft. in standard air, the actual

altitude of the lower side of the stratosphere varies from about 4 miles at the poles to 9 miles at the equator.

16. Relation of Pressure to Elevation for Isothermal Gases.— In studying the relation of pressure to elevation in compressible fluids it is necessary to take into consideration the change of density resulting from change of altitude.

Figure 6 is a diagram showing the variation of pressure p with the altitude z. Equation (2) can be applied to a small difference in elevation dz, the cylindrical free body shown in Fig. 6 corresponding to the free body previously shown in Fig. 3. It should

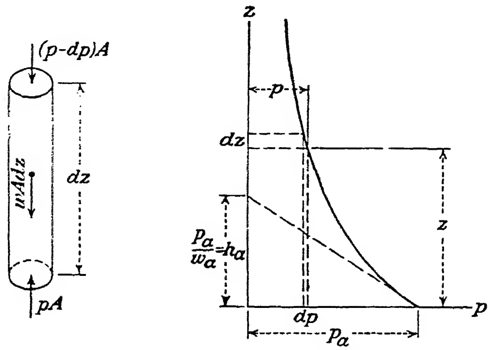


FIG. 6.—Pressure-elevation diagram for compressible fluids.

be noted that in Eq. (2) h is head, whereas in the following z is elevation, dp being a decrement. Then

$$dp = -w\,dz \tag{9}$$

and, by integrating between the pressure p at elevation z and pressure p_a at zero elevation,

$$p = p_a - \int_0^z w\,dz \tag{10}$$

The integration of the last term depends upon the way in which w varies with z, and it becomes necessary to find the pressure-elevation-density relationship. The development of this relationship for an isothermal gas follows.

According to Boyle's law, both the density and specific weight of a gas vary directly as the pressure for isothermal conditions, that is,

$$\frac{w}{p} = \frac{w_a}{p_a} \qquad \text{and} \qquad w = w_a\frac{p}{p_a} \tag{11}$$

Integrating Eq. (9) in the form $dz = -\frac{dp}{w}$, from pressure p_a to pressure p at a greater elevation,

$$z = \int_{p_a}^{p}\left(-\frac{dp}{w}\right) = -\frac{p_a}{w_a}\int_{p_a}^{p}\frac{dp}{p} = -\frac{p_a}{w_a}\log_e\frac{p}{p_a} = \frac{p_a}{w_a}\log_e\frac{p_a}{p} \qquad (12)$$

The expression p_a/w_a is sometimes called the atmospheric head h_a. It is the height of a hypothetical column of air of uniform weight w_a which will produce a pressure difference p_a. Physically it may be interpreted as the depth of a uniform atmosphere, with nothing above, which would produce atmospheric pressure p_a. Thus for standard air

$$h_a = \frac{p_a}{w_a} = \frac{2116 \text{ lb./ft.}^2}{0.07651 \text{ lb./ft.}^3} = 27{,}600 \text{ ft.}$$

By substituting any two elevations, z_1 and z_2, and the corresponding pressures, p_1 and p_2, in Eq. (12),

$$z_1 = h_a \log_e \frac{p_a}{p_1} \qquad \text{and} \qquad z_2 = h_a \log_e \frac{p_a}{p_2}$$

Then

$$z_2 - z_1 = h_a\left(\log_e\frac{p_a}{p_2} - \log_e\frac{p_a}{p_1}\right)$$

or

$$z_2 - z_1 = h_a \log_e \frac{p_1}{p_2} \qquad (13)$$

and

$$p_2 = p_1 e^{\frac{-z_2}{h_a}} \qquad (14)$$

The pressure at any elevation z in terms of pressure p_a at zero elevation becomes

$$p = p_a e^{-\frac{z}{h_a}} \qquad (15)$$

Equations (14) and (15) are useful in computing barometric pressures under isothermal conditions. Figure 6 is a graph of Eq. (15). The pressure approaches zero at great altitudes. The line cutting the z-axis at $z = h_a$ represents the pressure-elevation curve for an imaginary atmosphere of uniform weight w_a.

17. Relation of Pressure to Altitude with Temperature Gradient.—In making accurate computations of pressure or

density of gases over a wide range of altitude, the effect of the temperature gradient must be recognized. If the temperature gradient is $-\beta$ and the absolute temperature at zero elevation is T_a, then the absolute temperature at any elevation z is

$$T = T_a - \beta z \tag{16}$$

Returning to Eq. (9) and noting that $w = 1/v$,

$$dp = -w\,dz = -\frac{1}{v}\,dz \tag{17}$$

From the equation of state, Eq. (12), page 6, $pv = RT$ and $v = RT/p$. Putting this expression for v and the above expression [Eq. (16)] for T in Eq. (17)

$$\frac{dp}{p} = -\frac{dz}{RT} = -\frac{dz}{R(T_a - \beta z)} \tag{18}$$

Integrating between the pressure p at elevation z and pressure p_a at zero elevation,

$$\int_{p_a}^{p} \frac{dp}{p} = -\frac{1}{R\beta}\int_0^z \frac{dz}{\left(\frac{T_a}{\beta} - z\right)}$$

or

$$\left[\log_e p\right]_{p_a}^{p} = \frac{1}{R\beta}\left[\log_e\left(\frac{T_a}{\beta} - z\right)\right]_0^z$$

On substituting the limits, this becomes

$$\log_e \frac{p}{p_a} = \frac{1}{R\beta}\log_e \left(\frac{\frac{T_a}{\beta} - z}{\frac{T_a}{\beta}}\right) = \frac{1}{R\beta}\log_e\left(1 - \frac{\beta z}{T_a}\right)$$

or, finally

$$\frac{p}{p_a} = \left(1 - \frac{\beta z}{T_a}\right)^{\frac{1}{R\beta}} \tag{19}$$

For standard air $p_a = 2116$ lb. per sq. ft., $\beta = 0.00357$°F. per ft., $R = 53.3$ ft. per °F., and $T_a = 59° + 459.4° = 518.4$°F. Then

$$p = 2116\left(1 - \frac{0.00357z}{518.4}\right)^{5.256} \text{ (lb. per sq. ft.)} \tag{20}$$

Density and specific weight are proportional to pressure and inversely proportional to temperature. Therefore

$$\frac{w}{w_a} = \frac{\rho}{\rho_a} = \frac{p}{p_a}\frac{T_a}{T} = \frac{T_a}{T}\left(1 - \frac{\beta z}{T_a}\right)^{\frac{1}{R\beta}}$$

Substituting $T = T_a - \beta z$ from Eq. (16)

$$\frac{w}{w_a} = \frac{\rho}{\rho_a} = \left(1 - \frac{\beta z}{T_a}\right)^{\frac{1}{R\beta} - 1} \tag{21}$$

For standard air

$$= \rho g = 0.07651\left(1 - \frac{0.00357z}{518.4}\right)^{4.256} \text{ (lb. per cu. ft.)} \tag{22}$$

Equations (19) to (22) are valid up to the isothermal region or stratosphere. Pressure in the stratosphere can be computed by applying Eq. (19) up to the stratosphere and then using Eq. (14) over the remainder of the altitude. If z_s is the altitude at which the stratosphere begins and p_s is the corresponding pressure, then applying Eq. (19)

$$\frac{p_s}{p_a} = \left(1 - \frac{\beta z_s}{T_a}\right)^{\frac{1}{R\beta}} \tag{23}$$

In the stratosphere at a total altitude z the condition of constant temperature prevails and from Eq. (14) the corresponding pressure, where $h_s = p_s/w_s$, is

$$p = p_s e^{\frac{z_s - z}{h_s}} \tag{24}$$

Example.—Determine the temperature and pressure in an atmosphere having a temperature gradient of 0.00357°F. per foot at altitudes of 30,000 and 60,000 ft. if the stratosphere begins at 45,000 ft.

Solution.—At sea level standard atmospheric conditions are assumed: $t_a = 59$°F., $p_a = 2116$ lb. per sq. ft. abs., $\rho_a = 0.002378$ slug per cu. ft. Then designating conditions at 30,000 ft. and 60,000 ft. by the subscripts 1 and 2, respectively, the temperature at point 1 is obtained from Eq. (8*a*), which is $t = 59 - 0.00357z$. Hence

$$t_1 = 59 - 0.00357 \times 30{,}000 = 59 - 107 = -48\text{°F.}$$

The pressure as determined from Eq. (20) is

$$p_1 = 2116\left(1 - \frac{0.00357z}{518.4}\right)^{5.256} = 2116\left(1 - \frac{0.00357 \times 30{,}000}{518.4}\right)^{5.256}$$
$$= 2116(0.794)^{5.256} = 2116 \times 0.297 = 628 \text{ lb./sq. ft. abs}$$
$$= 4.36 \text{ lb./sq. in. abs.}$$

At the beginning of the stratosphere

$$t_s = 59 - 0.00357 \times 45{,}000 = -101.8°\text{F}.$$

and

$$p_s = 2116\left(1 - \frac{0.00357 \times 45{,}000}{518.4}\right)^{5.256} = 2116 \times 0.142$$
$$= 300 \text{ lb./sq. ft. abs.} = 2.09 \text{ lb./sq. in. abs.}$$

Point 2 is in the stratosphere where the temperature is constant so that $t_2 = t_s = -101.8°$F. The pressure at point 2 is given in terms of p_s by Eq. (14), that is,

$$p_2 = p_s e^{\frac{(z_s - z_2)}{h_s}}$$

From the equation of state,

$$h_s = h_a \frac{T_s}{T_a} = 27{,}600 \times \frac{357.6}{518.4} = 19{,}050 \text{ ft.}$$

Then

$$p_2 = 300(2.718)^{-\frac{15000}{19050}} = \frac{300}{2.20}$$
$$= 136.3 \text{ lb./sq. ft. abs.}$$
$$= 0.95 \text{ lb./sq. in. abs.}$$

Problem 18. What is the temperature in standard atmosphere at an altitude of 12,500 ft.? What is the temperature gradient in standard air in degrees centigrade if the altitude is measured in meters?

19. At what elevation in an isothermal atmosphere is the pressure 12 lb. per sq. in. abs.?

20. The pressure at a certain point in an isothermal atmosphere is 13 lb. per sq. in. abs. What is the pressure at a point 10,000 ft. higher?

21. What is the pressure in an atmosphere having the standard temperature gradient at altitudes of 15,000 ft. and 50,000 ft. and at the base of the stratosphere?

22. Compute the density of standard air at the three altitudes stated in Prob. 21.

18. Manometers.—The name manometer is given to a great variety of hydrostatic devices for indicating fluid pressure. The simplest of these is the open piezometer tube, Fig. 7*a*, which is merely an open tube connected to the container full of liquid. The height of the column of liquid in the open tubes above the point B is a measure of the difference between the pressure in the liquid at B and the atmospheric pressure on the surface at the top of the column. Using Eqs. (3) and (4), the pressure at point B is found to be

$$p = p_a + wh \text{ (on the absolute scale)}$$

or

$$p = wh \text{ (on the gage scale)}$$

The constants to be used depend upon the units in which it is desired to express the pressure. They are discussed in Art. 13.

In simple piezometer tubes the height $h = p/w$ and the pressure head at B may be instantly stated as h feet or inches of the liquid. The distance h in Fig. 7*a* is the same in all tubes and is independent of their shape or diameter except insofar as the diameter affects capillary action.

When a piezometer tube is connected to a container in which the pressure at B is less than atmospheric pressure, it must be in a position similar to that of the right-hand tube in Fig. 7*a*. The free surface in the tube will then be below the level of B in the container.

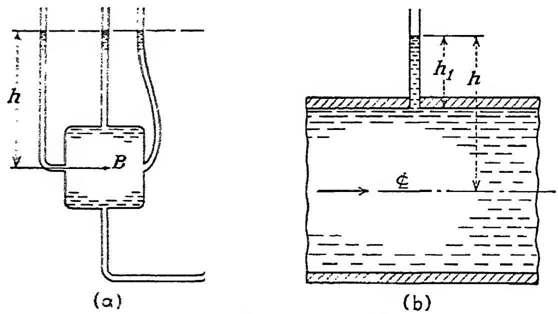


FIG. 7.—Piezometer tubes.

Figure 7*b* shows a piezometer tube connected to a pipe in which the liquid is moving. The height h_1 in this figure is a measure of pressure at the wall of the pipe if the opening is at right angles to the wall and free of any roughness or projection into the moving liquid. With these precautions the head h indicates the static pressure independent of the velocity, the pressure at the wall of the pipe being $p_1 = wh_1$ and that at the axis $p = wh$.

The manometers shown in Fig. 8 can indicate the pressure in either a gas or a liquid. Figure 8*a* is an example of a simple U-tube manometer in which a liquid of specific weight w_1 in the U-tube is used to indicate the pressure at B in another fluid of specific weight w. Starting with atmospheric pressure at the free surface in the open end of the tube and noting that pressure increases with depth, the resulting expression for pressure at B is

$$p_B \text{ (absolute)} = p_a + w_1h - wy \tag{25a}$$

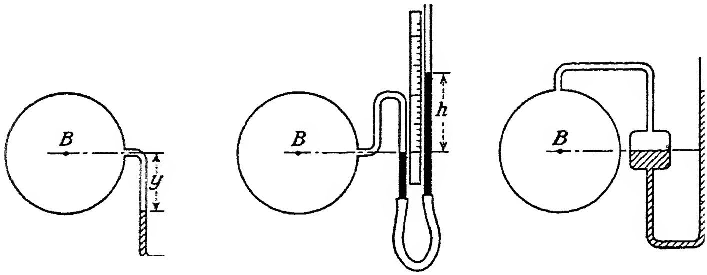
or

$$p_B \text{ (gage)} \quad w_1h - wy \tag{25b}$$

If the fluid at B is a gas, then w is very small as compared to w_1 and for most practical purposes Eqs. (25) become

$$p_B \text{ (absolute)} = p_a + w_1h \qquad \text{or} \qquad p_B \text{ (gage)} = w_1h \tag{26}$$

By making the lower portion of the tube flexible, as in Fig. 8*b*, the right side of the tube can be adjusted until the liquid in the left side is at the level of B. By this adjustment the term wy in Eqs. (25) is eliminated and they reduce to Eq. (26). This



(a) (b) (c)

FIG. 8.—Manometer tubes.

arrangement has the double advantage of placing the zero of the h scale in a fixed position and of having the scale read pressure or head in any desired unit independent of the density of the fluid at B.

The purpose of the flexible tube can be accomplished by the arrangement shown in Fig. 8*c*, which provides a reservoir for the indicating liquid in which the normal level is the same as point B. If the horizontal section of the reservoir is large as compared to the cross section of the tube along the scale, the level in the reservoir will change very little with small changes in pressure and in cases not demanding great accuracy the level may be considered constant. The zero of the scale is then in a fixed position and the height h on the scale indicates the head or pressure directly and Eq. (26) applies.

In the positions shown the manometers of Fig. 8 are indicating pressures greater than atmospheric. Any of them could also indicate pressures lower than atmospheric. In the latter case

h is measured in the opposite direction, and again proceeding from the free surface to B, the equations for Fig. 8b or 8c are

$$p_B \text{ (absolute)} = p_a - w_1 h \qquad (27a)$$

or

$$p_B \text{ (gage)} = -w_1 h \qquad (27b)$$

or

$$\text{vacuum} = w_1 h \qquad (27c)$$

Most of the equations in this and the following articles show the relation between pressure and manometer reading. The scale may be graduated so as to read directly in any unit of pressure or pressure head.

Liquids used in manometers are commonly chosen because they have a specific weight which gives a convenient height of column. Values of the specific gravity of commonly used manometer liquids are given in Table III.

TABLE III.—SPECIFIC GRAVITY OF MANOMETER LIQUIDS AT 20°C.

Liquid	Specific Gravity
Mercury	13.546
Acetylene tetrabromide	2.96
Alpha-chloronaphthalene	1.194
Ethyl alcohol	0.789
Toluene	0.866
Ethyl chloroacetate	1.159

It is desirable that liquids used in manometers leave the tube and fittings clean, that they be only slightly volatile and that they be noncorrosive to tubes and fittings. Where two fluids are in contact, they must be immiscible.

19. Differential Manometers.—The manometers discussed in the last article all have one end open to the atmosphere and all indicate the difference between the pressure of the atmosphere and that in the container. If the atmospheric pressure is known, the reading indicates an actual value of the pressure in the container.

There is a class of manometers not open to the atmosphere and therefore incapable of indicating an actual pressure. When connected to a fluid at two different points, they show the difference in pressure and are commonly called differential manometers or gages. Examples of these are shown in Figs. 9 to 12.

Figure 9 shows a tube connected to a pipe in such a way as to indicate the difference in pressures at M on the upstream side of a reducer and N on the downstream side. The specific weight of the fluid in the pipe is w and that of the manometer liquid is w_1. Beginning at point M and proceeding through the tube to N, always increasing pressure with depth, the equation for the system is

$$p_M + wy + wh - w_1h - wy = p_N$$

whence

$$p_M - p_N = (w_1 - w)h \tag{28}$$

Thus the difference in pressure is indicated by the height of the liquid column h and is independent of y. That is to say, the manometer may be placed any reasonable distance above or below points M and N without changing the reading, provided that the tube is full of fluid. When points M and N are not at the same level, Eq. (28) must be rewritten to include the difference in level. If the flowing fluid in Fig. 9 is a gas, its weight w is very small as compared with w_1 and can often be neglected, in which case Eq. (28) becomes

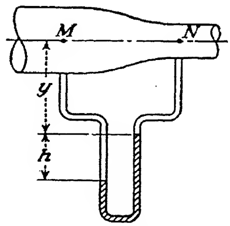


FIG. 9.—Differential manometer.

$$p_M - p_N = w_1h \tag{29}$$

If the flowing fluid is water and the tube contains mercury (specific gravity, 13.6), Eq. (28) becomes

$$p_M - p_N = (13.6w - w)h = 12.6wh \tag{30}$$

The difference in pressure heads, measured in feet of water, is

$$\frac{p_M}{w} - \frac{p_N}{w} = 12.6h \tag{31}$$

20. Micromanometers.—In some work, especially that dealing with aerodynamics, it is necessary to read pressures or pressure differences with great accuracy. The micromanometer is a special manometer designed to attain great accuracy or sensitivity.

Sensitivity of the devices shown in Figs. 7 and 8 can be increased by using a light liquid and thereby increasing h. If the liquid in the tube of Fig. 9 has a specific gravity only a little

greater than unity and the fluid in the pipe is water, the reading h from Eq. (28) will be very large and the manometer will be very sensitive.

A combination of two reservoirs and two liquids of slightly different density may be used as shown in Fig. 10. The reservoirs contain liquid of specific weight w_2 and in the lower portion of the U-tube there is a slightly heavier liquid of specific weight w_1. Spaces M and N above the free surfaces are connected by tubes to any two other containers in which the pressures are the same as at M and N. If the areas of horizontal cross sections of

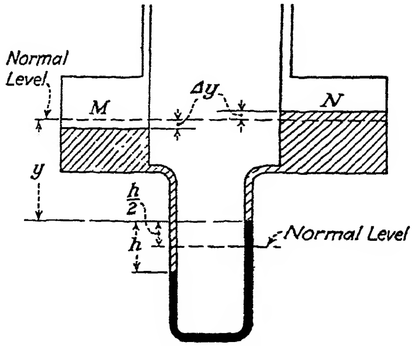


FIG. 10.—Sensitive differential manometer.

the reservoirs and the tube are A and a, respectively, then the displacement of the column in either tube is accompanied by the fraction a/A times as much displacement in the reservoir. Thus

$$\Delta y = \frac{ah}{2A}$$

Beginning with the pressure at M and proceeding through the tube to N, the following equation results:

$$p_M + w_2\left(y - \frac{ah}{2A}\right) + w_2h - w_1h - w_2\left(y + \frac{ah}{2A}\right) = p_N$$

$$p_M - p_N = (w_1 - w_2)h + \frac{a}{A}w_2h \qquad (32)$$

This type of gage has the advantage of giving a large reading for small pressure differences and also of entirely balancing any effect of capillary action. The last term of Eq. (32) may be

neglected in many cases, depending upon the value of the ratio a/A and the accuracy desired.

In the manometer shown in Fig. 11, the inclined tube is employed to gain sensitivity. It has a reservoir of area A and an inclined tube of cross section a. The scale is parallel to the tube, the reading on it being larger than that on a vertical tube in the ratio y/h, with a corresponding increase in sensitivity. Spaces M and N are connected to two points for which a difference

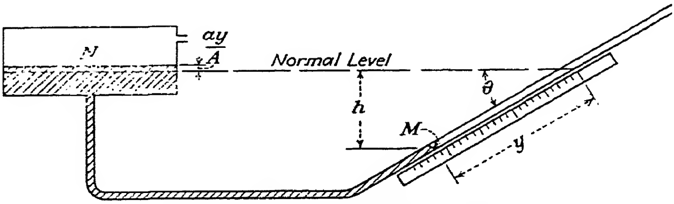


FIG. 11.—Inclined-tube manometer.

in pressure is wanted. The ratio of areas being a/A, the surface at N is raised $\frac{a}{A}y$ for a scale reading of y. Starting with the pressure at M and following pressure changes through the tube to N, neglecting the weight of gas in spaces M and N,

$$p_M - w_1 y \sin\theta - w_1 \frac{a}{A} y = p_N$$

from which

$$p_M - p_N = w_1 y \left(\sin\theta + \frac{a}{A}\right) \tag{33}$$

In terms of the slope of the tube, m or $\tan\theta$, and the diameters of the tube and reservoir, d and D, respectively, Eq. (33) takes the form

$$p_M - p_N = w_1 y \left(\frac{m}{\sqrt{1 + m^2}} + \frac{d^2}{D^2}\right) \tag{34}$$

Manometers of this type intended for accurate work, are equipped with spirit levels, leveling screws, reading glass and a means for adjusting the inclination of the tube. A photograph of such a device used in the University of Michigan wind tunnel is shown in Fig. 12.

Such things as varying density of liquid, weight of neglected gas columns, nonuniform diameter of tubes and varying effects of capillarity tend to prevent the attainment of extreme accuracy

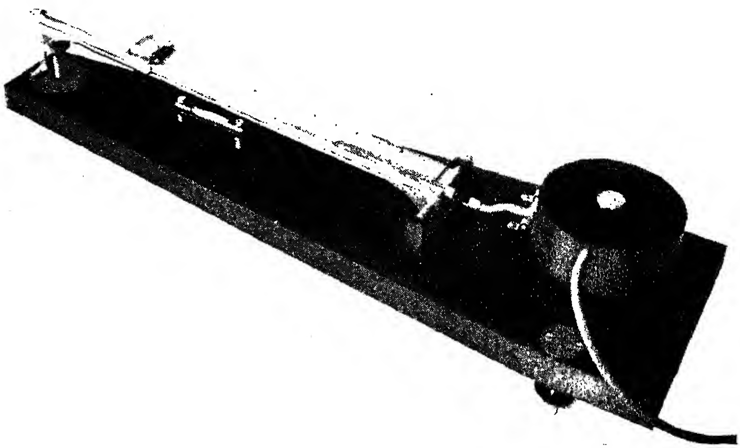
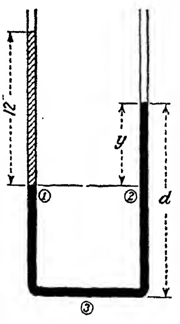


FIG. 12.—Inclined-tube manometer.

and make it necessary to calibrate this instrument and other accurate manometers for the conditions under which they are used.

Example.—A U-tube having an internal cross section of 0.25 sq. in. contains mercury to a depth of 10 in. What is the pressure at the bottom? What is the pressure after 3 cu. in. of acetylene tetrabromide is added to one leg of the tube?



Solution.—The specific gravity of mercury is 13.6 so that its specific weight is 13.6 × 62.4 lb. per cu. ft. Then the pressure at a depth of 10 in. or $^{10}/_{12}$ ft. is

$$\frac{10}{12} \times 13.6 \times \frac{62.4}{144} = 0.833 \times 13.6 \times 0.433 = 4.91 \text{ lb./sq. in. gage}$$

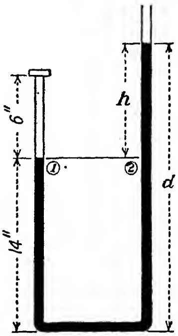
Adding the acetylene tetrabromide, specific gravity of 2.97, its depth is 3 ÷ 0.25 = 12 in., and the mercury stands as in the figure. The pressures or pressure heads at points 1 and 2 must be equal so that

$$2.97 \times 12 = 13.6y \qquad \text{and} \qquad y = 2.62 \text{ in.}$$

The depth d is now $10 + \dfrac{2.62}{2} = 11.31$ in. and at point 3 the pressure is

$$p_3 = \frac{11.31}{12} \times 13.6 \times 0.433 = 5.55 \text{ lb./sq. in. gage.}$$

Example.—A U-tube 20 in. high is filled with mercury to a depth of 12 in. One end is then closed and mercury is added to the other until it is 14 in. deep in the closed end. How deep is the mercury in the open end?



Solution.—If the cross section of the tube is A sq. in., then the initial volume of air in the closed end is $8A$ cu. in. and the new volume is $6A$. Then since $pv = C$, $p_1v_1 = p_1(6A) = 14.7(8A)$ and $p_1 = \frac{4}{3}(14.7)$. Since points 1 and 2 are at the same level in the mercury, it follows that $p_1 = p_2$. Expressing the latter in terms of head of mercury, h in.,

$$p_2 = \frac{4}{3}(14.7) = 14.7 + 13.6(0.433)\frac{h}{12}$$

and

$$h = 10.0 \text{ in.}$$

The final depth of mercury in the open tube is $14 + 10.0 = 24.0$ in.

Problem 23. In Fig. 7*a* the pressure at B is 2 lb. per sq. in. gage and the liquid is water. Compute h.

24. In Fig. 7*a* the liquid is oil, specific gravity is 0.8 and the pressure at B is 10 lb. per sq. in. abs. Where does the liquid stand in the right-hand tube, the other tubes being closed?

25. In Fig. 7*b* the pressure at the center of the pipe is 20 lb. per sq. in. gage. Compute h if the fluid is water.

26. In Fig. 8*a* the liquid at B is water and that in the tube has a specific gravity of 3. If $y = 18$ in., compute h for $p_B = 4$ lb. per sq. in. gage and 11 lb. per sq. in. abs.

27. The pressure at B of Fig. 8*b* is 5 lb. per sq. in. gage. The liquid is water and the tube contains mercury. Compute h. If the pressure is now increased by 2 lb. per sq. in. and the flexible tube is not adjusted, compute the new h measured from the zero of the scale.

28. The liquid in the tube of Fig. 8*c* has a specific gravity of 3 and B contains gas. What length of scale corresponds to a change of pressure of 1 lb. per sq. in. (*a*) if the reservoir is very large, (*b*) if the reservoir diameter is three times the diameter of the tube?

29. In Fig. 9 the pipe contains water. In what ratio is the gage reading, h, magnified by changing from mercury to a gage liquid having a specific gravity of 3?

30. In Fig. 10 spaces M and N contain air, the reservoirs a liquid with specific gravity of 0.8 and the tube a liquid with specific gravity of 1.2. The diameters of the reservoirs and tube are 3 in. and ¼ in., respectively. Compute h for a difference in pressure head of 1 cm. of water.

31. In Fig. 11 the angle θ is 10 deg. and the diameters of the tube and reservoir are 5 mm. and 10 cm., respectively. What percentage of error is introduced if the rise in the reservoir level is neglected when spaces M and N contain air? Is the percentage of error the same if these spaces contain a liquid?

21. Barometers.—The barometer is a simple hydrostatic device used to measure atmospheric or barometric pressure. If a tube closed at the upper end is filled with liquid and then placed as shown in Fig. 13*a*, the liquid will stand in the tube at a height depending on the atmospheric pressure on the free surface and the pressure in the tube. If all the air could be removed, the only pressure in the tube would be the vapor pressure necessary to balance the vapor tension of the liquid. Then proceeding from the closed space and through the liquid to the free surface, the equation

$$p_v + wh = p_a \qquad (35)$$

is obtained, in which p_v is the vapor pressure. Writing this in the form

$$h = \frac{p_a}{w} - \frac{p_v}{w} \qquad (36)$$

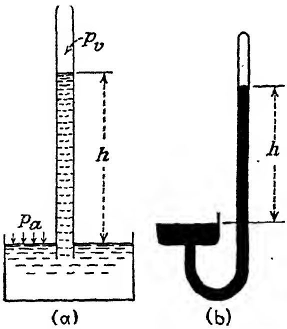


FIG. 13.—Barometers.

it is seen that the scale reading is reduced by the vapor pressure to the extent of p_v/w. For this reason it is important that the liquid used in such a barometer be one with a low vapor pressure and, to make the height small and easily readable, it should also be heavy. Mercury best satisfies these requirements and is the liquid commonly used. A typical mercury barometer tube is shown in Fig. 13*b*.

For standard air a barometer containing water would stand at a height

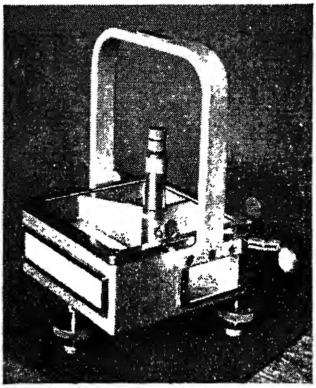
$$h = \frac{p_a}{w} - \frac{p_v}{w} = \frac{2116}{62.4} - \frac{p_v}{w} = 33.91 - \frac{p_v}{w} \text{ (ft.)}$$

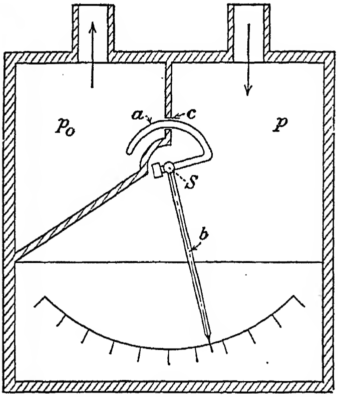
This value of h represents the maximum height of water column which can be supported by atmospheric pressure and it is therefore the maximum theoretical suction lift of a pump. Since the density varies with temperature, it is necessary to adjust both terms of Eq. (36) for high temperatures. Values of the vapor pressure for water are given in Table II, page 8. The column in a mercury barometer in standard air stands at a height of 29.92 in. or 760 mm.

Problem 32. How high is the column of a water barometer at 100°F. when the atmospheric pressure is 14.1 lb. per sq. in.?

33. A barometer has a column of mercury 28 in. high covered with 2 in. of water. If the temperature is 90°F., what is the atmospheric pressure?

22. Pressure Gages.—The use of manometers is limited to the measuring of pressures which produce a practical and readable length of column. In measuring high pressures or very small pressures it is necessary to use some other form of pressure gage. Such gages usually contain some elastic element which, when deformed by the pressure, has its motion transmitted by a mechanism to a pointer or dial. The elastic element may be a





(a) (b)

FIG. 14.—Reichardt pressure gage.

diaphragm, a bellows or sylphon, a Bourdon tube, a piston working against a spring, or a wire in torsion.

The torsion principle is used in a gage developed by Reichardt,[1] Fig. 14*a*, which is sensitive to pressures of the order of 10^{-5} mm. of water.

This gage, shown in plan in Fig. 14*b*, has two chambers connected to the points for which the pressure difference, $p - p_0$, is desired. The wall between the chambers has a small round opening *c*. A small glass piston *a*, in the form of a circular arc, passes freely through the opening and is suspended by a vertical wire at *S*. The wire is subjected to torsion when the piston moves and the pointer *b*, which is also suspended by the wire, moves along the scale. There is some flow around the piston

[1] REICHARDT, H., Druckmesser für sehr kleine Druckunterschiede, *z. Ver. deut. Ing.*, Band 79, p. 1503, 1935.

and through the opening which is corrected for in the calibration of the instrument.

23. Hydrostatic Devices.—Hydrostatic devices have many interesting applications in engineering and industry. One of these, the King-Seeley Telegage, Fig. 15, has a wide use for measuring depths, volumes or weights of liquids. It consists of a hydrostatic gage or manometer connected to a chamber C immersed in the liquid. This chamber is open at the bottom, and by pumping air into tube D the liquid level in the tube is forced down until air escapes into the tank A. The air pressure in the chamber is then the same as the liquid pressure at the same level in the tank and is also equal to the pressure on the surface of the manometer fluid in reservoir B. The scale having its index at the level of the free surface in B will then measure the pressure at the bottom of the tank or, when connected as shown, the difference in pressures at the surface in the tank and in the air chamber. By considering the horizontal section of the tank and weights of fluid and manometer fluid, the Telegage may be made to indicate depth, volume or weight in any desired units.

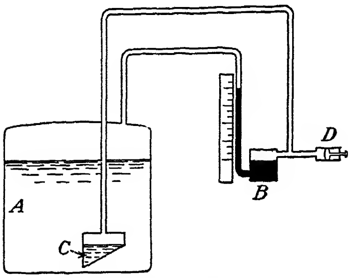


FIG. 15.—The King-Seeley Telegage.

A variation of the U-tube is sometimes employed both in running levels and measuring vertical distances. A device used at the General Motors Proving Ground for taking close vertical measurements of automobiles is shown in Fig. 16. It consists of a reservoir B connected by flexible tubing to manometer C. The reservoir is placed in a level position on the part to be measured and the liquid in the manometer comes to the same level as that in the reservoir. Since the total quantity of fluid is constant, the depth of the fluid in the reservoir varies. The scale of the manometer is adjusted to compensate for this change in depth so that elevation is read directly. Thus, if the horizontal section of the reservoir is A and the cross section of the tube is a, the divisions of the scale representing an elevation of 1 in. will have a length of $\left(\dfrac{A}{A+a}\right)$ in. The zero reading, which

varies with temperature, may be adjusted by expanding or contracting a sylphon which is connected to the base of the manometer.

A device[1] used in the wind-tunnel laboratory of the Daniel Guggenheim Airship Institute at Akron is shown in Fig. 17.

Fig. 16.—Hydrostatic device for taking vertical measurements. (*Courtesy General Motors Proving Grounds.*)

It consists of a closed reservoir A completely filled with liquid and connected by a tube such as B to a manometer. The force to be measured is transmitted by a wire T to the flexible diaphragm head of the reservoir. The diaphragm carries an adjustable initial load L, the variation in total load corresponding to the variation in tension in wire T. Any change in position of

[1] Troller, T., The Vertical Wind Tunnel of the Daniel Guggenheim Airship Institute, Daniel Guggenheim Airship Institute, *Pub.* 1, pp. 11–22.

the diaphragm is shown in the reading on the manometer scale, which, after careful calibration, is a measure of the tension in T.

24. Pressure Forces on Surfaces.—In Art. 11 it was shown that the pressure exerted on any surface by a static fluid is normal to that surface. The pressure force acting on area dA is $p\,dA$ and, the pressure on all elements of a plane area being in the same direction, the total pressure force is $\int p\,dA$. In gaseous fluids

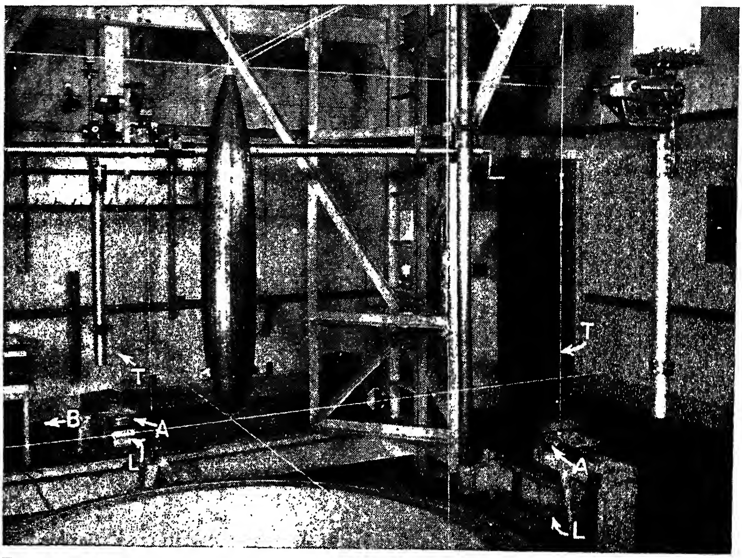


FIG. 17.—Hydrostatic device for measuring force. (*Courtesy Dr. T. Troller.*)

the pressure is practically constant over areas of ordinary size and therefore the resultant pressure force is

$$P = p\int dA = pA \tag{37}$$

This equation also holds for a horizontal area subjected to liquid pressure because the pressure on such an area is uniform.

Figure 18 represents a submerged plane area MN at an angle θ with the free surface of a liquid. It is desired to find the resultant pressure force exerted by the liquid on one side of the area. The pressure at depth h being $p = wh$, the pressure force dP on any element of area is

$$dP = p\,dA = wh\,dA$$

or, in terms of y, the distance measured along the surface from its intersection O with the free surface of the liquid,

$$dP = w(y \sin \theta)dA$$

The resultant pressure force P is

$$P = \int dP = w \sin \theta \int y\, dA$$

From the definition of a centroid, $\int y\, dA = \bar{y}A$, where $\bar{y}$ is the centroidal distance; since the depth of the centroid is $\bar{h} = \bar{y} \sin \theta$,

$$P = w(\bar{y} \sin \theta)A = w\bar{h}A \tag{38}$$

In Eq. (38) the product $w\bar{h}$ is the pressure at the centroid of the area. Thus it is apparent that the resultant pressure force

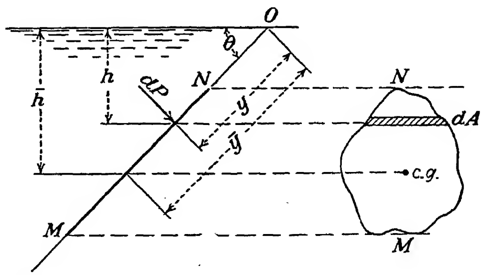


FIG. 18.—Resultant pressure force on a plane area.

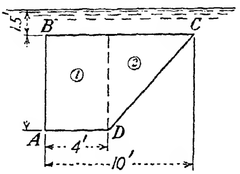
on a plane area is the pressure at the centroid multiplied by the area and also that the average pressure on a plane area is equal to $w\bar{h}$, the pressure at the centroid.

It will be noticed that the above discussion is based entirely upon gage pressure, atmospheric pressure being neglected. If the area is considered as closing an opening with the same atmospheric pressure on the dry side and on the surface of the liquid, then the real pressure force on an element of the wetted area is $(p_a + wh)dA$ and that on the dry side is $p_a\, dA$. Thus the effective pressure force is seen to be

$$(p_a + wh)dA - p_a\, dA = wh\, dA = dP$$

just as used in the development of Eq. (38), and it is obviously permissible to neglect atmospheric pressure whenever both sides of the area in question are affected by the same atmosphere.

Example.—Compute the pressure force exerted by water on the submerged vertical plane area $ABCD$.



Solution.—The centroid of the area is found to be 2.57 ft. below BC. The head or depth of water at the center of gravity is $1.5 + 2.57 = 4.07$ ft. and the pressure force is

$$P = w\bar{h}A = 62.4(4.07)(42) = 10{,}680 \text{ lb.}$$

If preferred the area can be divided into parts 1 and 2, for which the centroidal positions are well known. It is then unnecessary to find the centroid of the whole area since the pressure force may be written as

$$\begin{aligned} P &= P_1 + P_2 = w\bar{h}_1A_1 + w\bar{h}_2A_2 \\ &= 62.4(4.5)(24) + 62.4(3.5)(18) = 10{,}680 \text{ lb.} \end{aligned}$$

Problem 34. Compute the pressure force exerted by water on the following plane areas: (a) a vertical circle 4 ft. in diameter with its horizontal diameter 5 ft. below the free surface, (b) a vertical equilateral triangle with 6-ft. sides having one side in the water surface. (c) a rectangle 4 by 6 ft. making an angle of 60 deg. with the water surface and having a 4-ft. edge in the surface, (d) a vertical 4- by 4-ft. area with the upper edge 3 ft. below the water surface.

25. Center of Pressure.

—If the pressure on a plane area is uniform, the resultant pressure force passes through the centroid of the area. This will be the case only if the area is horizontal or when the pressure is that of a gas and can be considered as practically uniform. For areas not horizontal, it is often required to find the point at which the resultant pressure force intersects the area. This point is commonly called the center of pressure.

Since the elementary forces on the plane are all in the same direction, the process of finding the center of pressure is merely that of finding the position of the resultant of a system of parallel forces as it is done in statics.

Since the pressure force on an element of area dA (Fig. 19) is dP, the resultant force is $P = \int dP$ applied at a distance y_p from the intersection of the area with the water surface. The moment of P about this intersection is Py_p and the moment of dP around the same line is $y\,dP$. Equating the moment of the resultant force to the sum of the moments of all its parts,

$$Py_p = \int y\,dP \tag{39}$$

This may be considered a fundamental equation and it often affords the simplest and most direct method of finding y_p. In

setting up the equation, y and y_p may be measured from any line parallel to the free surface and the plane. Letting

$$dP = wh\,dA = wy \sin\theta\,dA$$

and $P = \int dP$, Eq. (39) becomes

$$y_p w \sin\theta \int y\,dA = w \sin\theta \int y^2\,dA$$

or

$$= \frac{\int y^2\,dA}{\int y\,dA} = \frac{I}{A\bar{y}} \tag{40}$$

in which I is the moment of inertia about the axis from which y is measured and $A\bar{y}$ is the moment of the area about the same axis.

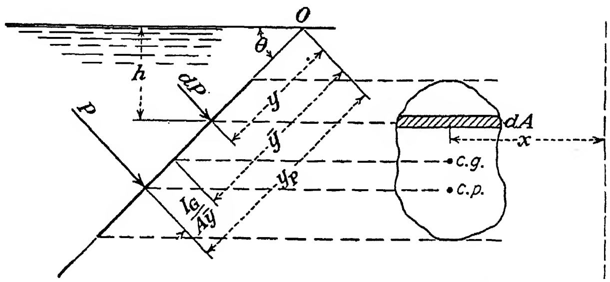


FIG. 19.—Center of pressure on a plane area.

The moment of inertia about the origin O may be expressed in terms of moment of inertia about a parallel axis through the centroid of the area as $I_G + A\bar{y}^2$. Substituting this value in Eq. (40)

$$y_p = \frac{I_G + A\bar{y}^2}{A\bar{y}} = \bar{y} + \frac{I_G}{A\bar{y}} \tag{41}$$

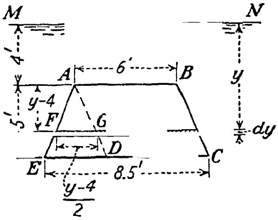
In many problems Eq. (41) furnishes a convenient means for finding y_p. It is also important because an inspection of it shows that the center of pressure is always below the center of gravity by the distance $I_G/A\bar{y}$ measured along the plane. The value of $I_G/A\bar{y}$ decreases with depth, that is, the center of pressure comes closer to the center of gravity as the depth is increased.

The relative horizontal position of the center of pressure on an area can often be determined by inspection. If necessary, the principle of moments may again be utilized, taking moments

about an axis in the plane and normal to its intersection with the free surface. The moment equation is then

$$Px_p = \int x\, dP$$

Example.—Find the center of pressure on the vertical trapezoidal surface shown in the figure.



Solution.—Begin with a horizontal element of area at a depth of y and having a width dy. Draw the line AD parallel to BC. Then the distance ED is 2.5 ft. and from the geometry of the figure $FG = (y - 4)/2$. Then the length of the element of area is $\frac{y-4}{2} + 6 = 4 + \frac{y}{2}$. The area of the element is

$$dA = \left(4 + \frac{y}{2}\right)dy$$

and the force acting on it is

$$dP = wy\, dA = w\left(4y + \frac{y^2}{2}\right)dy$$

The moment of dP around MN is $y\, dP$ and, dividing the total moment by the total force,

$$y_p = \frac{\int y\, dP}{\int dP} = \frac{\int_4^9 \left(4y^2 + \frac{y^3}{2}\right)dy}{\int_4^9 \left(4y + \frac{y^2}{2}\right)dy} = 6.95 \text{ ft.}$$

Problem 35. Compute values of y_p starting from Eq. (39) and check results by Eq. (41): (*a*) for Fig. 20*a*, (*b*) for Fig. 20*b*, (*c*) for Fig. 20*c*.

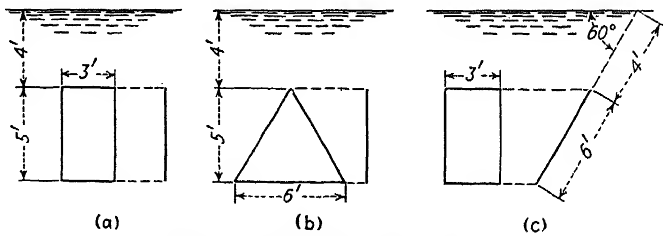


FIG. 20.—Submerged plane areas.

36. The center of pressure of a 4- by 9-ft. rectangle having the 9-ft. edge vertical is 5 ft. below the upper edge. At what depth is the upper edge?

37. Using Eq. (39) prove that (*a*) for a rectangle with one edge in the liquid surface, y_p = two-thirds of the altitude; (*b*) for a triangle with the vertex in the liquid surface and the base parallel thereto, y_p = three-fourths of the altitude; (*c*) for a triangle having one edge in the liquid surface, y_p = one-half of the altitude.

38. A trapezoid 5 ft. wide has parallel sides 6 ft. and 10 ft. long. Where is the center of pressure (*a*) when the 6-ft. edge is in the water surface, (*b*) when it is 4 ft. below the surface and the trapezoid is vertical?

26. Pressure Distribution and Pressure Volume.—A graphical concept of pressure distribution and pressure force is often helpful. Figure 21 shows diagrams of the variation of pressure with

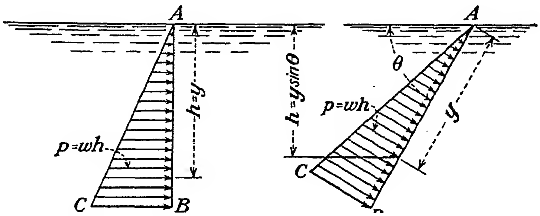


FIG. 21.—Variation of pressure with depth.

depth, in which AB is a plane surface and the ordinate extending from AB to AC is the pressure, $p = wh$, at any depth h.

The pressure distribution on an area is shown in Fig. 22. The rectangle $ABCD$ is an area subjected to hydrostatic pressure, the amount and direction of which are represented at any point by an ordinate perpendicular to the area and extending to surface $MNOP$. The pressure force on dA, an element of area in surface $ABCD$, is $p\ dA$. This expression is also an element of volume, dV, extending from surface $ABCD$ to surface $MNOP$. Then the resultant pressure force on the area is

$$P = \int dP = \int p\,dA = \int dV = V \tag{42}$$

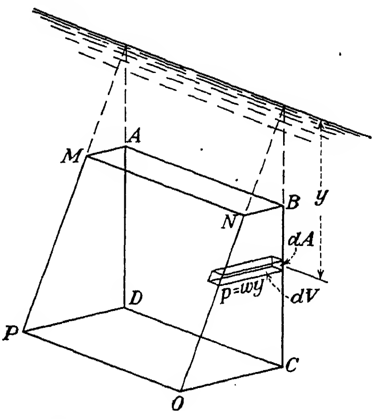


FIG. 22.—Pressure distribution and pressure volume.

in which V is the volume of the figure $ABCDPMNO$. The quantity V is called the pressure volume and will be expressed in some unit of force, as pounds or tons. The process of finding the position of the centroid of this pressure volume is the same

as that of finding the position of the resultant pressure on area $ABCD$, since

$$Py_p = \int y\,dP = \int y\,dV = V\bar{y}_V \tag{43}$$

and it follows that the center of pressure and center of the pressure volume are at the same distance from plane $ABNM$ or any parallel plane.

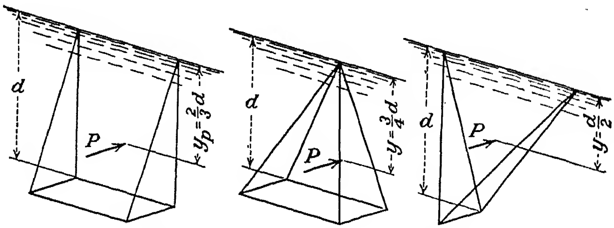


FIG. 23.—Typical pressure volumes.

An inspection of the pressure volume will often readily reveal the position of the center of pressure. Three common areas with their pressure volumes are shown in Fig. 23. The pressure volume for a rectangle with one edge in the free surface is a wedge, and it is apparent that the center of pressure is at two-thirds of the total depth. The pressure volume for a triangle with its vertex in the free surface and its base parallel to the surface is a pyramid, and from the geometry of the figure the center of pressure is at once seen to be at three-fourths of the total depth. The pressure volume for a triangle with one edge in the water surface is a tetrahedron, its center of gravity, and therefore the center of pressure, being at a depth equal to one-half of the total depth.

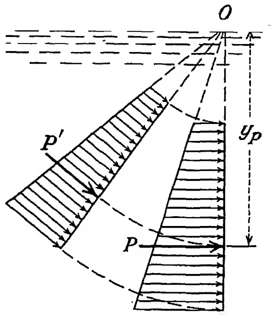


FIG. 24.—Effect of rotation on the center of pressure.

When any plane area is rotated about the intersection of its plane with the free surface, as in Fig. 24, the center of pressure remains in a fixed position on the plane. This is evident because all pressure ordinates are reduced in exactly the same proportion. This fact is also apparent from Eq. (40) or (41).

27. Pressure Forces on Curved Surfaces.—The determination of pressure forces on curved surfaces sometimes involves extensive calculations. The problem may often be simplified by giving attention to the components of pressure force rather than to the force itself. In Fig. 25, $ABCD$ represents a curved surface of area dA. For convenience it will be taken to have its radius of curvature R in a plane parallel to the xy-plane and at angle θ with a parallel to the y-axis. The area is subject to fluid pressure

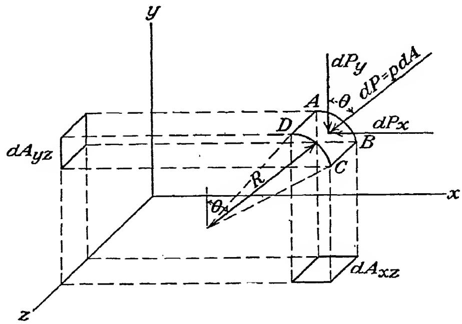


FIG. 25.—Pressure on a curved area.

p and the pressure force is $dP = p\,dA$ in the direction of R. The x- and y-components of the pressure force are $dP_x = p\,dA \sin\theta$ and $dP_y = p\,dA \cos\theta$. The products $dA \sin\theta$ and $dA \cos\theta$ are the projections of area dA on the yz- and xz-planes, respectively, whence

$$dP_x = p\,dA \sin\theta = p\,dA_y \tag{44}$$
$$dP_y = p\,dA \cos\theta = p\,dA_z \tag{45}$$

It may be stated as a principle that the component in any direction of pressure force on an area having only elementary dimensions is equal to the pressure force on the area projected on a plane normal to the component. When the pressure is uniform, the principle holds for an area of any size.

Figure 26 shows a curved area $ABCD$ of finite dimensions, subject to the pressure of a liquid, the free surface intersecting the curved surface along line AB. The pressure force on an element of area dA is $dP = wh\,dA$. The horizontal component of this force is

$$dP_x = wh\,dA \sin\theta = wh\,dA_{yz}$$

and the resultant horizontal force is

$$P_x = \int dP_x = w\int h\, dA_{yz} = w\bar{h}A_{yz} \tag{46}$$

in which $\bar{h}$ is the head on the centroid of the projected area, A_{yz}.

The horizontal component of the resultant pressure force on area $ABCD$ is the pressure force on its projection on the yz-plane, that is, $A'B'C'D'$, and the amount and position of this component are found in exactly the same way as for a plane area.

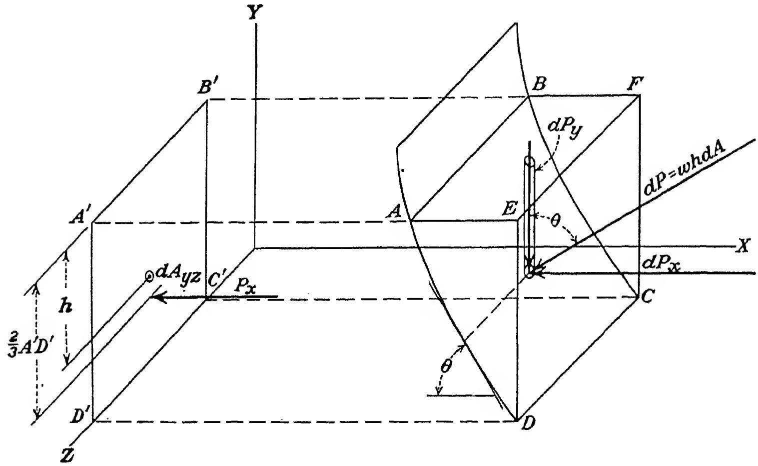


FIG. 26.—Pressure on a curved area.

The vertical component of dP is

$$dP_y = wh\, dA\ \cos\theta = wh\, dA_{xz}$$

The expression $wh\, dA_{xz}$ is the weight of a vertical prism of liquid extending from dA to the free surface and may be written $w\, dV$ where $dV = h\, dA_{xz}$. Then letting V be the total volume of liquid above the curved area, the vertical component of the resultant pressure force is

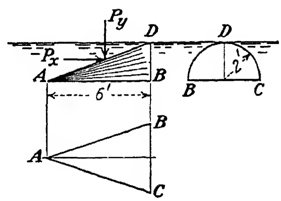
$$P_y = \int dP_y = w\int h\, dA_{xz} = w\int dV = wV \tag{47}$$

The product wV is the total weight of liquid in the volume $ABCDEF$ lying above the curved area. Thus the resultant vertical pressure force is caused by and equal to the weight of this volume and it can be demonstrated that it acts vertically through the centroid of the volume. For the sake of convenience the

area used in this demonstration has been taken as a cylindrical area with its elements normal to the xy-plane. The principle involved in finding components of pressure force on a warped surface is no different except that for such a case a component of force parallel to the z-axis is introduced.

If the lower side of area $ABCD$ is subject to pressure of a liquid having its free surface at the same level as in this discussion, the components of pressure force will be found in the same way but will be opposite in direction.

Example.—In the figure one-half of a right circular cone is just submerged in water with the triangular face horizontal. Compute the horizontal and vertical components of pressure force, P_x and P_y, respectively, on the curved surface.



Solution.—The component P_x is the same as the pressure force on the projection of the area, which in this case is the semicircle BDC. The area is then $\pi r^2/2 = 6.28$ sq. ft., the center of gravity is at a distance

$$\frac{4r}{3\pi} = 0.848 \text{ ft.}$$

from the diameter and the head on the center of gravity is

$$\bar{h} = 2 - 0.848 = 1.152 \text{ ft.}$$

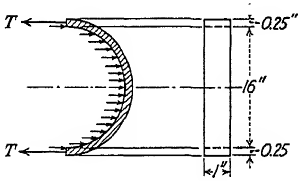
Then

$$P_x = w\bar{h}A = 62.4 \times 1.152 \times 6.28 = 452 \text{ lb.}$$

The vertical component P_y is the weight of the water above the surface. In this case the volume is that of a triangular prism less the half cone and the force is

$$P_y = [(12 \times 2) - (\tfrac{1}{3} \times 6.28 \times 6)]62.4 = 714 \text{ lb.}$$

Example.—A 16-in. steel pipe carries air at 200 lb. per sq. in. What is the circumferential tensile stress in the wall of the pipe if it is 0.25 in. thick?



Solution.—Consider a portion of the pipe 1 in. long. The resultant pressure force on any half of such a section is equal to the pressure times the projected area, which in this case is 16 sq. in. Then

$$P = 16 \times 200 = 3200 \text{ lb.}$$

and the tension in each side is $T = P/2 = 1600$ lb. The stress is

$$T/\text{area} = 1600 \div 0.25 = 6400 \text{ lb. per sq. in.}$$

Problem 39. With what force is a hemispherical suction cup 1 in. in diameter and having no air under it held against a windshield?

40. An opening in the bottom of a tank is closed by a hemisphere 4 ft. in diameter with the convex surface uppermost. If water stands 2 ft. above the crest of the hemisphere, with what pressure force is it held against the bottom?

41. A cone 2 ft. in diameter and 3 ft. in altitude is held vertex downward with its base at the water surface. What is the resultant pressure force on the conical surface?

42. Fig. 27 shows a partial section and end view of a tank containing water. Find (*a*) the vertical component of pressure force on *BC*, (*b*) the horizontal component of pressure force on *BCD*, (*c*) the resultant pressure force on *BC*.

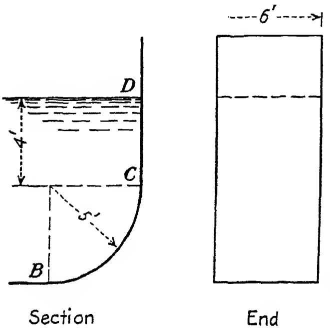


FIG. 27.

General Problems

43. Compute the pressure at *B* in Fig. 28, (*a*) when the fluid at *B* is air, (*b*) when it is water. If the pressure at *B* is now increased by 5 lb. per sq. in., what is the new difference in elevation of the mercury surfaces (*c*) with air at *B*, (*d*) with water at *B*?

44. Compute the difference in pressures at *M* and *N* of Fig. 29. The pipe contains water and the bottom of the U-tube contains mercury.

45. In Fig. 15 the liquids in the tank and manometer have specific gravities of 1.20 and 3.00, respectively. The pressure on the surface inside of the tank is 4 lb. per sq. in. and the opening of the chamber *C* is 5 ft. below the surface. What is the reading of the manometer in inches?

46. The pressure at a given point on a gas main is indicated by a 2-in. column of water in a U-tube open to air at one end. If the gas weighs 0.035 lb. per cu. ft. and the air is standard, what is the reading of a similar U-tube at a point 400 ft. higher under static conditions?

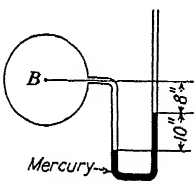


FIG. 28.

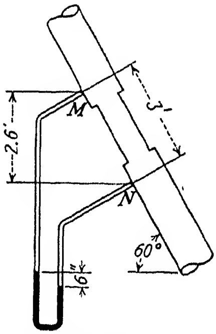


FIG. 29.

47. A cubical tank measuring 6 ft. on each side is filled with water. Find (*a*) the pressure force on the side, (*b*) the pressure force on the bottom, (*c*) the depth at which a horizontal line must be drawn to divide the pressure force on a side evenly, (*d*) the center of pressure on the lower half of one side.

48. A cylindrical barrel 1 ft. in diameter and 3 ft. long is fitted with a 1-in. pipe extending 50 ft. above the barrel. The barrel lies on its side and the staves are held by four hoops. If the pipe is filled with water, what is the average tensile force in the hoops due to water pressure?

49. A right triangle having a base of 4 ft. and altitude of 6 ft. is immersed in water with its base horizontal and the 6-ft. side vertical. Find both coordinates of the center of pressure (*a*) with the vertex in the water surface, (*b*) with the vertex 3 ft. below the surface.

50. The gate, a cross section of which is shown in Fig. 30, is 5 ft. wide. Find the pressure force and the center of pressure (*a*) on the right surface of the gate, (*b*) on the left surface of the gate; (*c*) find the vertical force at *D* to open the gate, neglecting its weight.

51. What is the resultant of the pressure forces on the two sides of the gate of Fig. 30?

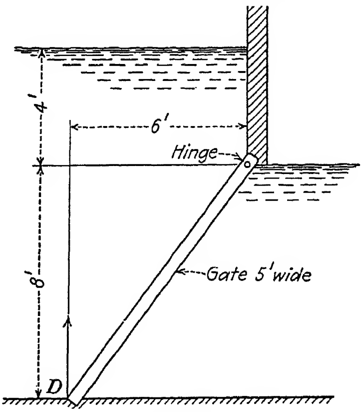


FIG. 30.

52. A box measuring 4 ft. on each edge is half filled with liquid of specific gravity of 2, the upper half being filled with water. Find (*a*) the pressure force on a side of the box, (*b*) the center of pressure on a side, (*c*) the center of pressure on the lower half of a side.

53. A square area having diagonals 12 ft. long is submerged in water. Find the total pressure and locate the center of pressure when one diagonal is vertical and ends in the free surface.

54. A submerged circular area 6 ft. in diameter is in a vertical position and tangent to the water surface. Where is the center of pressure? At what depth is the center of the area if the center of pressure is 3 in. below the center?

55. In Fig. 31 a circular opening is closed by a sphere. The pressure at *B* is 50 lb. per sq. in. abs. What horizontal force is exerted by the sphere on the opening?

FIG. 31.

56. A right cone having a base 2 ft. in diameter and an altitude of 3 ft. has its axis vertical and base uppermost and is filled with water. What is the horizontal pressure force on one-half of the internal area? If the same cone is held submerged in the same position with its base in the water surface, what is the upward force on the outside of the cone? What is the horizontal force on one-half of the outer surface?

57. A vertical tube of uniform diameter and 8 ft. long is open at the bottom and closed at the top. Originally full of air, it is submerged vertically in water until it is half full of air. To what depth is the top submerged?

58. In Fig. 32 a pressure of 164.7 lb. per sq. in. gage in cylindrical space *B* forces the movable piston *D* to the right, thereby reducing cylindrical space

C which is filled with oil. The fixed piston E is perforated by a $\frac{1}{4}$-in. tube. What is the pressure in line F leading to a closed space?

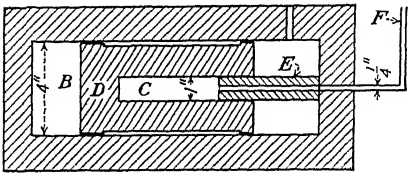


FIG. 32

59. A spherical balloon is 10 in. in diameter when inflated to a pressure of 2 lb. per sq. in. gage. What is the tensile force in the walls per inch of circumference?

60. The gate shown in cross section in Fig. 33 is 10 ft. long. The curved face BC is a circular arc. Find (*a*) the moment about the hinge of the pressure force on the gate, (*b*) the vertical and horizontal components of pressure force on BC, (*c*) the resultant pressure force on BC.

61. Find the depth of the center of pressure on Fig. 34 when the area is vertical. How far is the center of pressure from AB if the surface makes an angle of 30 deg. with the vertical?

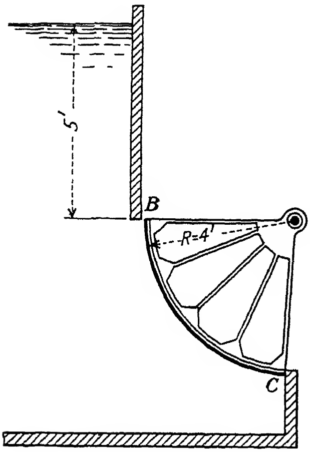


FIG. 33.

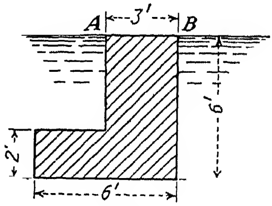


FIG. 34.

62. A closed airtight box is 4 ft. square and 8 ft. high. It is filled to a depth of 6 ft. with water, the pressure in the air space above being -0.866 lb. per sq. in. gage. What is the effective pressure force on a side of the tank?

CHAPTER III

FLOTATION

28. Buoyant Force or Static Lift.—Archimedes stated the principle that a body immersed in a fluid is buoyed up or supported by a force equal to the weight of fluid displaced. The truth of this principle may be demonstrated by considering the pressure forces on a submerged body.

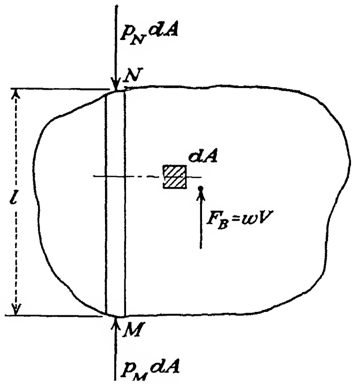


FIG. 35.—Buoyant force or static lift.

Figure 35 represents a body completely surrounded by any fluid. It may be divided into vertical prisms such as MN. Using the principle of projected areas discussed in the previous articles, the upward vertical component of pressure force on the lower end of the prism is $p_M\, dA$ while the downward component of pressure force on the top is $p_N\, dA$. Then the difference between the upward and downward pressure forces is

$$p_M\, dA - p_N\, dA = (p_M - p_N)dA = wl\, dA = w\, dV$$

in which dV is the volume of the elementary prism MN. Thus the resultant vertical pressure force on the prism is the weight of fluid displaced by the prism, and the sum of all such elementary forces is

$$F_B = \int w\, dV = wV \tag{1}$$

The product wV is the total weight of fluid displaced, that is, the "displacement" of the body. It is the buoyant force or static lift. It acts vertically through the center of gravity of the displaced fluid, which point is commonly called the center of buoyancy. While it is convenient to determine the amount of

static lift by computing displacement, it is often helpful to note that it is merely the resultant vertical pressure force.

A body floating at a plane of separation between two fluids presents the commonest problem in buoyancy, the usual fluids being air and water. Let Fig. 36 represent a body submerged partly in a fluid of specific weight w_2 and covered by a lighter fluid of specific weight w_1. The difference in upward and downward pressures on the vertical element of volume dV is

$$(p_M - p_N)dA = (w_1l_1 + w_2l_2)dA = w_1l_1\,dA + w_2l_2\,dA = w_1\,dV_1 + w_2\,dV_2$$

dV_1 and dV_2 being, respectively, the portions of dV displacing fluid of specific weights w_1 and w_2. The total static lift is

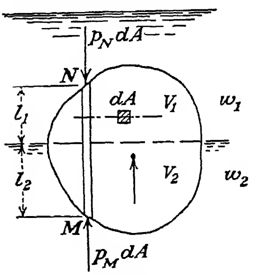


FIG. 36.—Static lift in two fluids.

$$F_B = \int w_1\,dV_1 + \int w_2\,dV_2 = w_1V_1 + w_2V_2 \qquad (2)$$

which is again the total weight of fluid displaced.

In the case of bodies floating at the free surface of a liquid, w_1 is usually neglected because it is very small as compared to w_2. The static lift is therefore only the weight of displaced liquid.

The horizontal pressure forces on an immersed body can be investigated by dividing it into horizontal prisms. The axial components of pressure force on the two ends of any such prism are equal and opposite and, their sum being zero for every elementary prism, there can be no resultant horizontal pressure force for the body as a whole when the fluid is static.

Problem 63. A sphere 2 ft. in diameter floats half submerged in water when free. What force is needed to hold it completely submerged?

64. A body weighing 400 lb. and having a specific gravity of 0.4 is held submerged in water by an anchor weighing 150 lb. per cu. ft. What is the minimum weight of the anchor?

65. A hemisphere 4 ft. in diameter is under water with the flat side against a vertical surface, all fluid being excluded between them. What is the buoyant force? The water is 6 ft. deep and the hemisphere rests on the bottom with all fluid excluded. What central force is required to lift it?

66. A cylindrical body 6 in. in diameter and 2 ft. long is designed to float half above the plane of separation of two liquids which have specific gravities of 1.25 and 1.05. What is the weight of the body?

67. A box 4 ft. square by 6 ft. long outside has sides and one end 6 in. thick. It is submerged in water with the open end down, the closed end flush with the water surface, and it is half filled with air. What is the static lift on the box?

68. A body weighing 56 lb. per cu. ft. floats in water having a specific gravity of 1.04. If the volume above water is 10 cu. ft., what is the total volume?

29. Equilibrium of Floating Bodies. In discussing flotation or equilibrium the word displacement will be used not only in the previously employed sense to indicate weight of displaced liquid but also in the sense of change of position. Thus linear displacement and angular or rotational displacement will designate change of position effected by linear or rotational movement from the position of equilibrium.

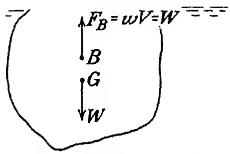


FIG. 37.—Stable equilibrium. Center of gravity below center of buoyancy.

A floating body in equilibrium is supported by a static lift which is equal, opposite to and in the same line of action as its weight. The equilibrium of the body is said to be stable if any change from its position, however small, is accompanied by the introduction of forces or moments tending to return it to its original position. The equilibrium will always be stable if the center of gravity of the floating body is lower than the center of buoyancy, as illustrated by Fig. 37. In order for the relationship between center of gravity and center of buoyancy shown by this figure to exist, it is necessary that the body be weighted or otherwise nonhomogeneous. It will remain in the position shown unless it is rotated by additional forces into another position of stable equilibrium and it will always recover from any slight linear or rotational displacement because such a displacement immediately introduces an unbalanced force or a stabilizing moment.

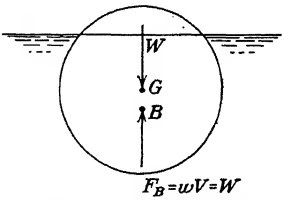


FIG. 38.—Neutral equilibrium.

A homogeneous cylinder or sphere having the center of gravity on its geometrical axis will float in a condition of neutral equilibrium, as shown in Fig. 38. It will remain in any given position unless displaced and will not recover from any rotational displacement. This may be seen by noting that the buoyant force is

the vector sum of the pressure forces, all of which are radial and therefore pass through the center of gravity. This being true, these forces, including weight, are concurrent, and the sum of the moments of all forces is zero. There can be neither a stabilizing nor an overturning couple and the equilibrium is said to be neutral.

30. Metacenter and Metacentric Height.—Stable equilibrium of floating bodies is not limited to cases in which the center of gravity is below the center of buoyancy. There are other stable conditions which are more readily studied by introducing the notion of metacenter and metacentric height.

Figure 39*a* represents one cross section of a floating body with the center of gravity and center of buoyancy of the entire body

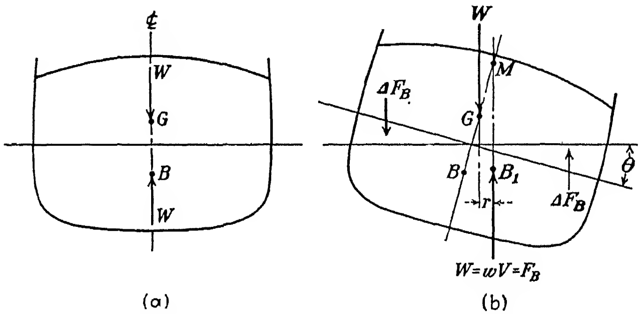


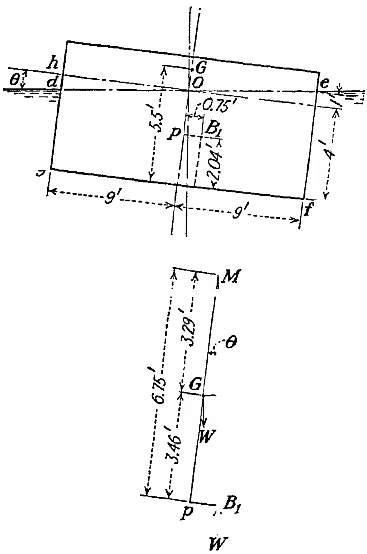
FIG. 39.—Metacenter and metacentric height.

at G and B, respectively. Although the center of gravity is above B, the equilibrium is stable if any rotation of the body from the position shown is accompanied by the introduction of forces tending to restore it to its original position with the axis vertical. If the body is rotated through a small angle to the position shown in Fig. 39*b*, the displacement is reduced by an amount ΔF_B on the left and increased by a like amount on the right. As a result of this change in shape of the displacement, the buoyant force is shifted to a new center of buoyancy, B_1, and it then intersects the extension of line GB at point M. For very small angular displacements point M is the metacenter of the floating body. When the metacenter lies above G, the weight W and the equal and opposite static lift F_B are in such relative positions that they constitute a couple tending to turn the body back to its position of equilibrium. If G is above M, there is no

righting couple and the force system is unstable. Then point M represents the limit above which G must not go, hence the term metacenter, which means limit center. Strictly speaking, it is the center of curvature of the path traversed by B at the instant the body starts to rotate.

The moment of the righting couple is Wr. For small angles of heel the position of M changes very little, the distance $m = \overline{MG}$ is nearly constant and the righting couple may be written

$$T = Wm\theta \qquad (3)$$



from which it is seen that the restoring moment is proportional to m. The metacentric height is then a measure of the stability and for that reason it is a very important property of a boat or any floating structure.

Example.—A scow having a uniform cross section 18 ft. wide has a draft of 4 ft. and its center of gravity is 5.5 ft. above the bottom. It is tipped to the position in the figure. Find the distance MG and the moment arm of the righting couple.

Solution.—The new center of buoyancy, B_1, is the centroid of the trapezoid *defg*. This is found to be 2.04 ft. from the bottom and 0.75 ft. from the axis of symmetry. The distance $Gp = 5.50 - 2.04 = 3.46$ ft. From similarity of triangles *hod* and B_1pM, $Mp/B_1p = ho/hd$ and

$$Mp = 9 \times 0.75 = 6.75 \text{ ft.}$$

Then

$$MG = 6.75 - 3.46 = 3.29 \text{ ft.}$$

The angle $\theta = \tan^{-1} \frac{1}{9} = 6°20'$. The perpendicular distance between W and the buoyant force is

$$Gq = MG \sin\theta = 3.29 \times 0.1103 = 0.363 \text{ ft.}$$

Problem 69. A square timber 6 ft. long and 12 in. square floats with 11 in. submerged and the sides vertical. If it is tipped so that one edge is in the water surface, what are the metacentric height and the righting couple?

31. Computation of Metacentric Height.—In dealing with a floating body such as a ship, the cross section varies from place

to place along the longitudinal axis in such a way that the positions of B_1 and M are not readily computed. Figure 40 shows a section and the water line of such a body whose longitudinal axis is ZZ_1 and whose cross section at the center of gravity is $DEFH$, the original water line on this section being LJ. It is displaced by rotation through a very small angle θ, the new water line being KN and the water surface $KZNZ_1$.

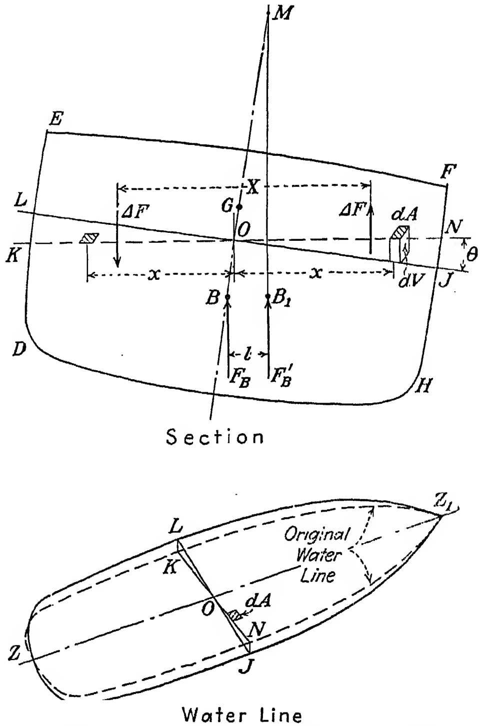


FIG. 40.—Computation of metacentric height.

In undergoing an angular displacement θ, the buoyant force has been changed by the addition of a wedge of displacement $NJZZ_1$ on the right and the loss of another wedge of displacement $KLZZ_1$ on the left. These wedges must be of exactly the same volume and it can be demonstrated that the line of intersection of the old and new water surfaces contains the center of gravity of the water plane for any shape of body when angle θ is small. Owing to the change in form of the volume of water displaced

by the body, the buoyant force is shifted from B, the original position, to B_1, the new center of buoyancy. In finding the position of B_1 it is convenient to form a notion of the new force F_B' acting through B_1 as consisting of:

a. The original buoyant force F_B.

b. An increment of upward force ΔF equal to the displacement of wedge $NJZZ_1$, acting at a distance $X/2$ to the right of ZZ_1.

c. A decrement of upward force ΔF equal to the displacement of wedge $KLZZ_1$, acting at the same distance to the left of ZZ_1.

Using this notion, it follows that

$$F_B' = F_B + \Delta F - \Delta F$$

or, in words, that the resultant is the sum of parts. Setting the sum of the moments of the forces in the second member equal to the moment of their resultant F_B' and adopting point B as the center of moments,

$$F_B'(l) = F_B(0) + (\Delta F)X$$

from which it is seen that the center of buoyancy has been shifted from B to B_1 through a distance

$$l = \frac{(\Delta F)X}{F_B'} = \frac{(\Delta F)X}{F_B} \tag{4}$$

The wedges of displacement form a couple of moment $(\Delta F)X$, which can be reduced to more convenient terms by considering the wedges to be made up of elements of volume such as dV. The length of the element is $x\theta$ and the horizontal section is dA so that its displacement in units of force becomes

$$dF = w\,dV = wx\theta\,dA$$

and the moment of the element of force about axis ZZ_1 is $x\,dF$. Then the sum of the moments of all such forces over the entire water section, $KZNZ_1$, about axis ZZ_1 or any parallel axis, is

$$\int x\,dF = w\int x\,dV = w\theta\int x^2\,dA = w\theta I \tag{5}$$

in which I is the moment of inertia of the entire water section of the body about axis ZZ_1. By substituting this expression for the moment of the couple in Eq. (4)

$$l = \frac{(\Delta F)X}{F_B} = \frac{w\theta I}{wV} = \frac{\theta I}{V} \tag{6}$$

From the geometry of Fig. 40, $l = \theta\overline{MB}$ and it follows that

$$\overline{MB} = \frac{I}{V} \tag{7}$$

In words, this equation states that the distance $\overline{MB}$ for a small angular displacement is equal to the moment of inertia of the water section of the body divided by the volume below the water section. The metacentric height of a body for small angles of heel is then

$$m = \frac{I}{V} - \overline{GB} \qquad \text{or} \qquad m = \frac{I}{V} + \overline{GB} \tag{8}$$

depending upon whether the center of gravity is, respectively, above or below B. If the body is wall-sided, that is, with fairly vertical sides, the metacentric height is nearly constant for angles up to several degrees. It appears that m will be smallest for the smallest possible value of I, that is, when I is taken about the axis of minimum moment of inertia, and that the body will be least stable against rotation about that axis. It is also apparent that the body is most stable, that is, it has the maximum metacentric height, when rotation is considered to be about the axis through the centroid of the water section for which I is maximum.

When the metacentric height as obtained from Eq. (8) is negative, that is, when M is below G, there is no righting couple, but on the contrary an overturning one, and the equilibrium is unstable.

32. Floating Vessel Containing Liquid.—In the previous article it was shown in Eq. (6) that when a floating body is rotated through a small angle θ the center of buoyancy is moved transversely through a distance $l = \frac{w\theta I}{wV} = \frac{\theta I}{V}$. In the case of a floating vessel containing liquid with a free surface (Fig. 41), it can be shown by a method paralleling that used in the last article that the center of gravity G of the load is shifted transversely through a horizontal distance

$$q = \frac{w'\theta I'}{wV} \tag{9}$$

in which w' is the specific weight of the liquid in the vessel and I' is the moment of inertia of the free surface within the vessel about the axis through O' normal to the cross section shown.

Under these conditions the righting couple consists of a vertical force $W = wV$ acting downward through G_1 and H and an equal upward force acting through B_1. The distance between the

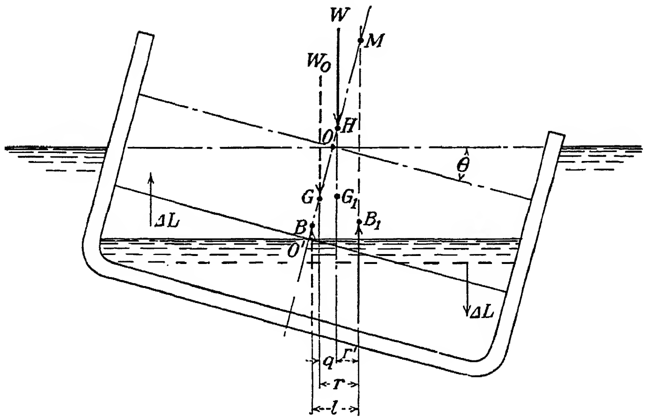


FIG. 41.—Floating vessel containing liquid.

forces constituting the couple is reduced by the distance q so that the new arm r' is

$$r' = r - q$$

and the effective metacentric height is reduced from $\overline{MG}$ by the distance $\overline{HG} = q/\theta = w'I'/wV$. The new metacentric height m' is then $m' = \overline{MH} = r'/\theta$ and for the case illustrated

$$m' = \overline{MH} = \overline{MB} - \overline{HG} - \overline{GB}$$

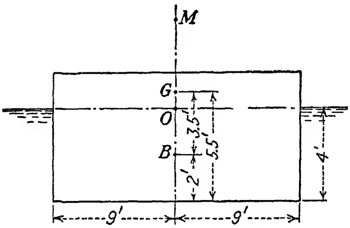
or

$$m' = \frac{I}{V} - \frac{w'}{w}\frac{I'}{V} - \overline{GB} \tag{10}$$

A comparison of Eqs. (10) and (8) shows that the presence of a free liquid surface impairs the metacentric height by the amount of $w'I'/wV$. The stability is reduced by a corresponding amount. The effect of the shifting load can be decreased by dividing the vessel with bulkheads and thereby decreasing I', in which case I' becomes the sum of the moments of inertia of all the free surfaces about their respective axes.

Example.—A scow 60 ft. long has a uniform cross section 18 ft. wide and draws 4 ft. of water. The center of gravity is 5.5 ft. above the bottom. Find the metacentric height and check against the distance MG as found in the example on page 51.

Solution.—The horizontal section at the water surface is a rectangle 18 ft. by 60 ft. Its moment of inertia about the longitudinal axis through O is



$$I = \frac{bd^3}{12} = \frac{60 \times 18^3}{12} = 29{,}160 \text{ ft.}^4$$

and the volume of water displaced is

$$V = 60 \times 18 \times 4 = 4320 \text{ cu. ft.}$$

From Eq. (8) the metacentric height is

$$\frac{I}{V} - GB = \frac{29{,}160}{4320} - 3.5 = 3.25 \text{ ft.}$$

This is to be compared with $MG = 3.29$ ft. as obtained for the same scow by another method in the last example. It appears that the difference is only about 0.04 ft. when the angle of heel is 6 deg.

Problem 70. A timber b ft. square and l ft. long is assumed to float in water with the sides vertical. Compute the metacentric height when (*a*) specific gravity is 0.16, (*b*) specific gravity is 0.5, (*c*) specific gravity is 0.84.

71. A caisson in the form of a cylinder closed at the bottom floats in water with the axis vertical. It is 24 ft. in diameter and floats with 18 ft. submerged. The center of gravity is on the axis and 7 ft. above the bottom. Compute the metacentric height. What is the limiting height of the center of gravity above the base for stability?

72. If the caisson has walls 2 ft. thick and there is 2 ft. of water inside, compute the metacentric height, other conditions being the same as in Prob. 71.

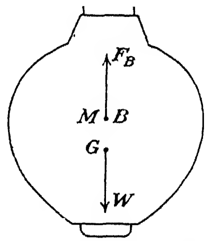


FIG. 42.—Submerged body.

33. Immersed Bodies.—In order for a body completely immersed in any static fluid to be in stable equilibrium, the usual conditions for stable vertical equilibrium must exist and the metacentric height must be positive in sign. The metacenter when immersion is complete is at the center of buoyancy B (Fig. 42), which for homogeneous fluids is the center of the volume of displacement. The metacentric height MG can then be positive in sign only when the center of gravity G is below B. If a rigid body in an absolutely incompressible liquid is in equilibrium at any depth, it will also be in equilibrium at all other depths. When the body is nonrigid or is of

varying displacement and when the medium in which it is immersed is of varying density, complicating factors are introduced.

The gas-filled balloon, Fig. 43, is an example of a nonrigid body in a compressible medium. It is usually permissible to consider the density of both the air and gas as constant throughout the height of the balloon, the variation being only a fraction of 1 per cent. It is never permissible to consider the pressure as constant through the height because, as heretofore explained, buoyant force is due entirely to pressure difference, and to eliminate that difference would be to eliminate completely all buoyancy or static lift. Conditions for equilibrium of a balloon are identical with those for a body submerged in liquid. For vertical stability the total displacement of air must equal the total weight of the balloon including its load and the gas with which it is inflated. Let w_a and w_g be the specific weights at a pressure of 1 atmosphere of air and gas, respectively, and let W be the total weight of balloon and load. Then, when the balloon is inflated to its full volume V, it is just in equilibrium if

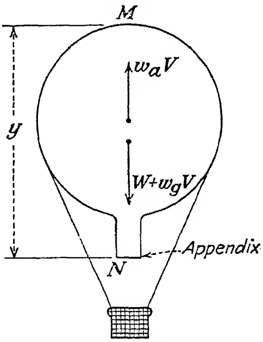


FIG. 43.—Equilibrium of a balloon.

$$w_aV = W + w_gV \qquad \text{or} \qquad W = (w_a - w_g)V \tag{11}$$

in which the expression $(w_a - w_g)V$ is the net static lift on the bag and the relatively very small lift on other parts is neglected. If this quantity is greater than W, the balloon is subjected to a resultant upward force of

$$R = (w_a - w_g)V - W \tag{12}$$

If the bag were entirely closed, the gas would tend to expand as the external pressure decreased with altitude and would eventually rupture the bag. With the appendix open at the bottom, the gas can escape as the balloon rises, and the internal and external pressures are just balanced at the open end and approximately so throughout. Letting the new specific weights of air and gas at any altitude be w_a' and w_g', respectively, with

the volume of the bag remaining practically constant, the resultant force at any altitude is

$$R = (w_a' - w_g')V - W = L - W \tag{13}$$

If the temperatures outside and inside of the bag are equal, the specific weights of gas and air are in the same ratio at all altitudes, that is, $w_g/w_a = w_g'/w_a'$. By writing the equation for net static lift on the bag in the form

$$L = (w_a' - w_g')V = \left(1 - \frac{w_g}{w_a}\right)w_a'V = Kw_a'V \tag{14}$$

it can be seen that L in this equation and in Eq. (13) is proportional to $w_a'V$, the weight of air displaced, for the condition of equal temperature for gas and air.

The external pressure and internal pressure being equal at the open end of the appendix, there will be an excess of internal pressure at all higher points, the excess being greatest at the summit where the valve is placed. The resultant static lift may then be adjusted either by discharging gas through the valve or by throwing out ballast.

If the pressure at point N is p_1 and the density is taken to be constant throughout the vertical distance y, the external pressure at M is

$$p_M \text{ (external)} = p_1 - w_a'y$$

and the internal pressure is

$$p_M \text{ (internal)} = p_1 - w_g'y$$

The difference between the external and the internal pressures, that is, the effective pressure on the surface of the bag at M, is

$$p' = (p_1 - w_g'y) - (p_1 - w_a'y) = (w_a' - w_g')y \tag{15}$$

As long as this is positive in sign, the bag is taut, but when it becomes negative, that is, when internal pressure in places is less than external pressure, the bag will become slack and the equations of this article are not applicable.

General Problems

73. A piece of wood weighs 4 lb. in air and a piece of lead weighs 4 lb. in water. The lead and wood together weigh 3 lb. in water. What is the specific gravity of the wood?

74. A pole 24 ft. long and 4 in. square has a specific gravity of 0.5. One end is in water and the other is suspended by a string which holds it 6 ft. above the surface. Find (*a*) the length of pole submerged, (*b*) the pull on the string and its direction.

75. A piece of timber having width *b* and depth *a* has a specific gravity of 0.5. What minimum ratio, b/a, will allow it to float in water with the sides vertical?

76. Figure 44 shows a thin-walled inverted box 5 ft. long and 1 ft. square, which was full of air before immersion. It is held in water in position (1) by 1 cu. ft. of concrete weighing 150 lb. Compute d_1. At what other depth, d_2, is the system in equilibrium. Is this a case of stable equilibrium?

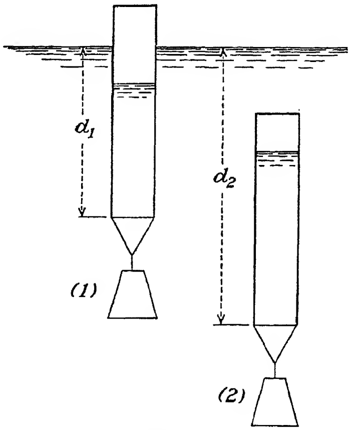


FIG. 44.

77. The solid timber raft shown in Fig. 45 floats half submerged when it has no load. Find (*a*) the position of the new water line on end *CDEF* if a load of 624 lb. is placed at the center of the raft, (*b*) the position of the new water line on end *CDEF* if a clockwise couple of 5616 ft.lb. in plane *ABCD* is exerted on the raft with no other load, (*c*) the position of the new water line if a load of 624 lb. is placed at the middle of *CF*.

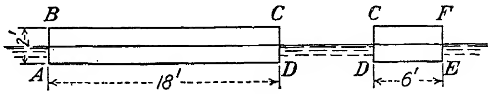


FIG. 45.

78. The raft in Fig. 45 is half submerged and is hinged along edge *B*. If the water rises 2 in., where is the water line on *CDEF*? With the water as shown, what load at the midpoint of *CF* will just submerge this edge?

79. A stick of timber weighing 40 lb. per cu. ft. is 20 ft. long and 1 ft. square. It would normally float on its side. How many steel plates 1 ft. square, ¼ in. thick and weighing 10 lb. each must be added to one end to make the timber float with the axis vertical?

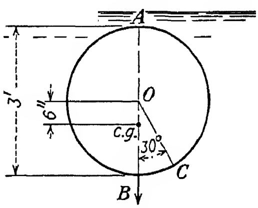


FIG. 46.

80. A submarine of 1200 tons displacement has its center of gravity 1 ft. below its center of volume. What is the righting couple when it is submerged in sea water (64 lb. per cu. ft.) and the angle of heel is 5 deg.?

81. The spherical buoy of Fig. 46 weighs 600 lb. It is held submerged in fresh water by an anchor attached at *B*. What is the righting couple on

the buoy when the axis AB is turned through 20 deg.? In what position is axis AB if the anchor line is attached at C?

82. A stick of timber 6 in. square and 24 ft. long weighs 22.4 lb. per cu. ft. Find the amount and direction of the force at one end to submerge 20 ft. of the timber in water. What is the position of the timber?

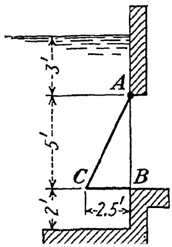


FIG. 47.

83. A log 1 ft. in diameter weighs 31.2 lb. per cu. ft. What is the shortest length that will float in water with the axis horizontal?

84. The gate, a cross section of which is shown in Fig. 47, closes an opening 5 ft. by 6 ft. The gate weighs 1200 lb., the center of gravity being 1 ft. from AB and BC. Compute the reactions at the hinge A and at B, assuming the latter to be horizontal.

85. Figure 48 shows a section through a cylindrical float. What load on the top will make it flush with the water surface?

86. A scow 20 ft. wide by 80 ft. long has a draft of 8 ft. The center of gravity is 5 ft. above the bottom. What is the metacentric height? Through what distance does the water line rise on the side if a 10-ton weight is shifted 15 ft. across the scow?

87. A homogeneous cylindrical float, 8 ft. long and 1 ft. in diameter, floats half submerged. What is the longitudinal metacentric height?

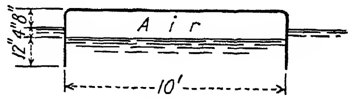


FIG. 48.

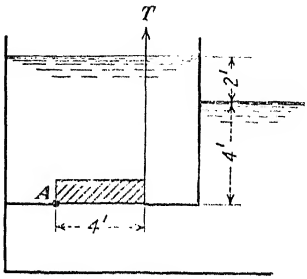


FIG. 49.

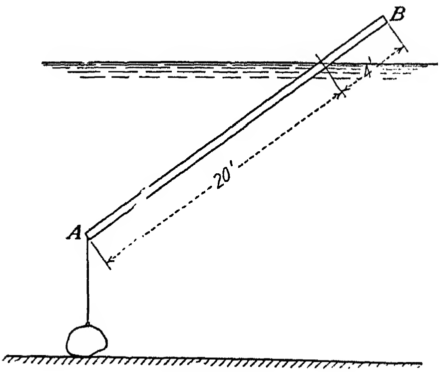


FIG. 50.

88. In Fig. 49 a gate 4 ft. square and 1 ft. thick is hinged at A. It is subjected to water pressure on both sides. What force T is required to open the gate if it weighs 400 lb.?

89. A solid float 1 ft. deep, 3 ft. wide and 8 ft. long floats with 2 in. in water and 6 in. in oil (specific gravity 0.8). What is the specific weight of the float? Is it more stable or less stable than it would be when floating in water?

90. The timber AB of Fig. 50 is 6 in. by 6 in. Find the specific weight of the timber and the total weight of the anchor if it weighs 150 lb. per cu. ft.

91. A balloon of 50,000 cu. ft. displacement is filled with hydrogen, which under the same conditions as standard air has a specific weight of 0.0069 lb. per cu. ft. Assuming the volume of the balloon to be constant, compute the static lift after ascending to an altitude of 10,000 ft. in standard air.

CHAPTER IV

ACCELERATED LIQUIDS IN RELATIVE EQUILIBRIUM

34. Forces on Fluids in Uniform Acceleration.—A static fluid has been previously defined as a body of fluid at rest or moving bodily at a uniform velocity. If a container of fluid is given a constant acceleration, the fluid, after adjusting its position or pressure distribution or both, comes to rest relative to the

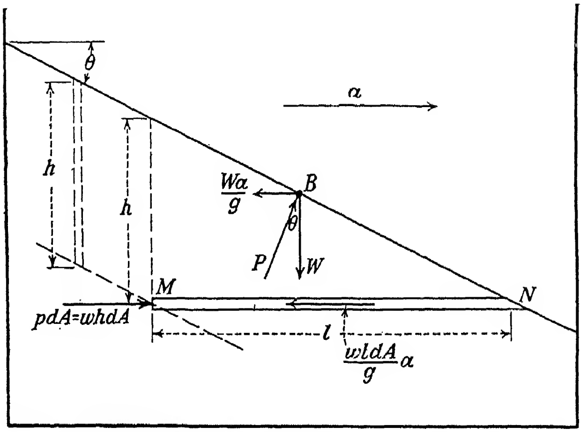


FIG. 51.—Liquid with uniform linear acceleration.

container and is said to be in a condition of relative equilibrium. It is then being accelerated as a body of constant form, and here, just as in static fluids, the body must be entirely free of any shear stress.

Figure 51 shows a vessel containing liquid which has a uniform acceleration of a toward the right. Considering a particle of fluid at B, it is found to be subject to two real forces, namely, the pressure force P of the surrounding liquid and the weight W of the particle. Since there is no shear force, P must be normal to the free surface. Following D'Alembert's principle, an imaginary force $\frac{W}{g}a$, called the reversed effective force or inertia force, may be added to the system, after which the system of three

forces must obey the ordinary laws of statics. Applying the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$, it is found that

$$W \tan \theta = \frac{Wa}{g} \qquad \text{or} \qquad \tan \theta = \tag{1}$$

From this equation the slope, $\tan \theta$, of the surface in relative equilibrium is seen to be the ratio of the horizontal acceleration to the acceleration of gravity.

When the acceleration is horizontal the vertical forces are the same as in a static liquid. Thus the pressure at distance h below the free surface is that required to support the weight of a vertical column of liquid and $p = wh$ as in static fluids, the pressure being constant in a plane parallel to the free surface.

Equation (1) may be verified by applying D'Alembert's principle to a horizontal prism MN having one end at the free surface and the other at a distance h below the surface. The length being l and the cross section dA, the weight is $wl\,dA$ and the reversed effective force is $\frac{wl\,dA}{g}a$. Applying the equation of statics $\Sigma F_x = 0$,

$$p\,dA = wh\,dA = \frac{wl\,dA}{g}a$$

whence

$$\frac{h}{l} = \frac{a}{g} = \tan \theta$$

which confirms Eq. (1).

35. Relative Equilibrium of Rotating Fluids.—If a container of liquid is rotated about its vertical axis, the motion is transmitted by shear stress to the liquid, which after a short time has the same angular velocity throughout as the container and is in equilibrium relative to it. The free surface is now curved in the form shown in Fig. 52. The principles used in the last article may be employed to study the form of the surface and the forces involved.

Any particle such as B, with an angular velocity ω about the axis OY and at a distance x therefrom, has a normal or centripetal acceleration of $\omega^2 x$. This acceleration and the effective force causing it always act toward the axis of rotation, and the

reversed effective force $\frac{W}{g}\omega^2 x$ acts away from the axis. Following D'Alembert's principle, this force and the real forces of pressure P normal to the surface and W acting vertically are

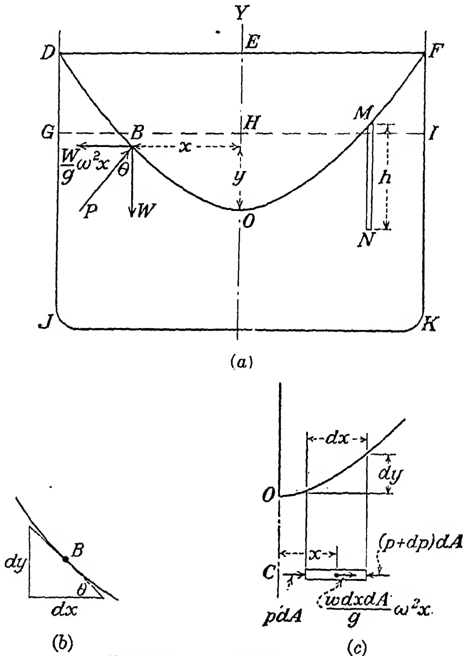


FIG. 52.—Rotating liquid.

placed on the particle. The laws of statics may now be applied to the particle and from $\Sigma F_x = 0$ and $\Sigma F_y = 0$

$$W \tan \theta = \frac{W}{g}\omega^2 x$$

in which θ represents the inclination of the surface as shown in Fig. 52b. Then

$$\tan \theta = \frac{\omega^2 x}{g} = \frac{dy}{dx} \tag{2}$$

whence, by integration, the constant of integration being found to be zero from conditions at O,

$$x^2 = \frac{2g}{\omega^2}y \qquad \text{or} \qquad y = \frac{\omega^2 x^2}{2g} \tag{3}$$

This is the equation of a parabola *DOF* with axis vertical and the origin of coordinates at *O* and the surface generated by its revolution is a paraboloid. It can be shown that the volume between the paraboloid and a horizontal plane through *DF* is one-half the volume of a circumscribed cylinder of diameter *DF* and height *OE*; the static free surface *GHI* is therefore midway between *O* and *E*.

It is noteworthy that, by letting the velocity of a rotating particle at radius x be $V = \omega x$, Eq. (3) may be written $y = V^2/2g$. The expression $V^2/2g$ occurs very frequently in hydraulics, where it is known as velocity head.

Because the acceleration is horizontal in direction, the vertical forces on any free body taken from the rotating liquid are the same as for static conditions. The pressure at the bottom of prism *MN* is then wh and any surface of equal pressure is a paraboloid parallel to the free surface. If the container holds two liquids having a surface of separation, that surface becomes a paraboloid parallel to the free surface.

Equation (3) may also be obtained by applying D'Alembert's principle to a free body of the rotating liquid in the form of a radial prism of length dx and cross section dA, Fig. 52*c*. The pressure force on the inner end is then $p\,dA$ and that on the outer end is $(p + dp)dA$; the reversed effective force is mass times acceleration or $\frac{w\,dx\,dA}{g}\omega^2 x$; and, letting $\Sigma F_x = 0$,

$$(p + dp)dA - p\,dA = \frac{w\,dx\,dA}{g}\omega^2 x \tag{4}$$

$$dp = \frac{w}{g}\omega^2 x\,dx = w\,dy \tag{5}$$

By integration

$$y = \frac{\omega^2 x^2}{2g} \tag{6}$$

which verifies Eq. (3) above.

Equations (4) and (5) are adaptable to the case of a gas rotating uniformly with its container. The horizontal prism of Fig. 52*c* may well be a free body of gas and Eq. (4) is correct as before. The specific weight is dependent upon the pressure p and at a given temperature $p = Cw$ from Eq. (1) of Chap. I. Substituting $w = p/C$ in Eq. (5),

$$\frac{dp}{p} = \frac{\omega^2}{Cg}x\,dx \tag{7}$$

Integrating this equation from pressure p_0 at the axis to p on the same level at any radius x,

$$p = p_0 e^{\frac{\omega^2 x^2}{2Cg}} \tag{8}$$

36. Hydrostatic Accelerometer.—A simple device for measuring accelerations which is based on the principles of relative equilibrium consists of a U-tube, Fig. 53, in which the difference in level of the two legs indicates the acceleration of the tube.

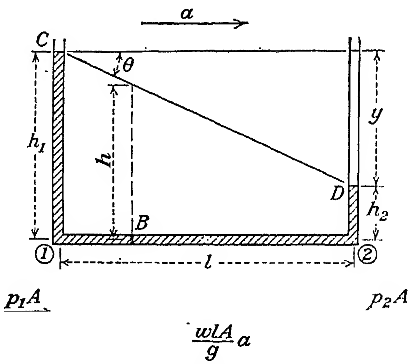


FIG. 53.—Hydrostatic accelerometer.

The liquid in the horizontal portion of the tube may be taken as a free body of length l and cross section A with pressure forces on each end. A reversed effective force equal to mass times acceleration is placed at the center of gravity and, applying D'Alembert's principle,

$$p_1A - p_2A = \frac{wlA}{g}a$$

Canceling A and expressing p in terms of head,

$$wh_1 - wh_2 = wl\frac{a}{g} = wy$$

Then

$$\frac{y}{l} = \frac{a}{g} = \tan\theta \qquad \text{or} \qquad a = \frac{gy}{l} \tag{9}$$

The reading of the instrument is seen to be the same for all liquids. The pressure head at any point B is the distance h measured from B to the pressure grade line CD. If a piezometer tube were inserted anywhere in the U-tube, the liquid would rise in it to line CD. In more complicated instruments various combinations of liquids and reservoirs may be used to magnify the readings.

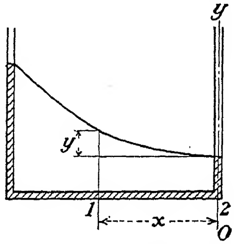


FIG. 54.—Rotating U-tube.

If the U-tube shown in Fig. 54 is rotated about axis Oy through one of the legs, the pressure grade line takes the form of a parabola, the equation of which is $y = \omega^2x^2/2g$, Eq. (3). This can be seen by referring to Eqs. (5) and (6) or by using a horizontal prism of unit cross section and length x as a free body; with the reversed effective force at the center of its length equal to mass times acceleration of the center, it appears that

$$p_1 - p_2 = \frac{wx}{g}\frac{\omega^2 x}{2} = wy \tag{10}$$

and

$$\frac{\omega^2x^2}{2g} = \frac{V^2}{2g}$$

If a U-tube is rotated about a vertical centroidal axis, the liquid does not change its position in the tube. The pressure gradient is a parabola ABC of Fig. 55 and, if the speed exceeds that at which the pressure at D becomes zero absolute, the column will break and a cavity will be formed at D.

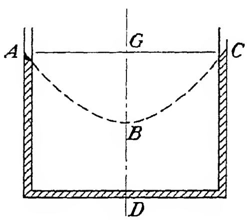
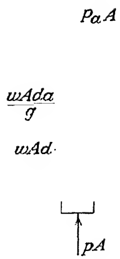


FIG. 55.—U-tube rotating about its central axis.

Example.—A tank of water 5 ft. deep is allowed to fall vertically with an acceleration of 16.1 ft. per sec.². What is the pressure on the bottom of the tank? At what acceleration is the pressure reduced to zero gage? What maximum acceleration can the tank have if the water is to remain in contact with the bottom?

Solution.—Consider a prism of water of horizontal cross section A extending to a depth d where the absolute pressure is p. The forces are then p_aA on the top, pA on the bottom and a weight equal to wAd. Adding the reversed effective force, $\frac{wAd}{g}a$, and writing $\Sigma F_y = 0$.



$$pA - p_aA - wAd + wAd\frac{a}{g} = 0$$

and

$$p = p_a + wd\left(1 - \frac{a}{g}\right)$$

Substituting $d = 5$ and $a = 16.1$, the pressure is

$$p = 2116 + 62.4 \times 5\left(1 - \frac{16.1}{32.2}\right)$$
$$= 2272 \text{ lb./sq. ft. abs.}$$

From inspection of the equation it is seen that the pressure is p_a absolute or zero gage throughout when $a = g$, that is, when the tank falls freely.

If the tank is given such an acceleration that the pressure at the bottom is zero absolute, the water starts to break away from the bottom and a space or cavity is formed. This is a simple case of the phenomenon known as cavitation. Setting $p = 0$ in the equation and solving for a,

$$0 = p_a + wd\left(1 - \frac{a}{g}\right)$$

and

$$a = g\left(\frac{p_a}{wd} + 1\right) = g\left(\frac{34}{d} + 1\right)$$

In this case,

$$= g(3\!4\!/\!5 + 1) = 7.8g = 251 \text{ ft./sec.}^2$$

37. Flotation in Accelerated Liquids.—In Chap. III it was pointed out that buoyant force or static lift is produced by and equal to the excess of upward pressure of the fluid on the body. The parabolic pressure grade line existing in rotating fluids suggests that there are cases in which the downward pressure force on a submerged body may exceed the upward, with an apparent reversal of direction of buoyancy. Such a case is illustrated by Fig. 56.

Let $ABCD$ be a tube closed at A and D and open only at E. The pressure at E being atmospheric, the pressure gradient during rotation is represented by the parabola FEG. Consideration will now be given to the pressure on a prism of liquid of length ds along the tube and cross section dA normal to it, lying at a distance x from the axis. Considering the equilibrium of this body along the axis of the prism to be affected only by forces having components parallel thereto, the forces involved are $p\,dA$ on the lower end, a larger pressure force $(p + dp)dA$ on the

upper end, the weight $W = w\,ds\,dA$ and the reversed effective force $\dfrac{W\omega^2 x}{g} = \dfrac{w\,ds\,dA}{g}\omega^2 x$. Taking components of these forces parallel to the tube, the equation of equilibrium is

$$(p + dp)dA - p\,dA = \frac{w\,ds\,dA}{g}\omega^2 x \cos\alpha - w\,ds\,dA \sin\alpha \qquad (11)$$

and

$$dp\,dA = W\left(\frac{\omega^2 x}{g}\cos\alpha - \sin\alpha\right) \qquad (12)$$

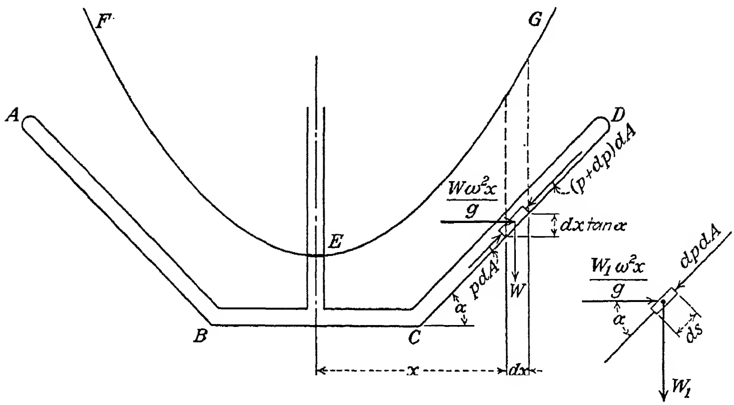


FIG. 56.—Flotation in rotating liquid.

or, canceling dA in Eq. (11) and setting $dx = ds \cos\alpha$,

$$dp = w\left(\frac{\omega^2 x}{g} - \tan\alpha\right)dx \qquad (13)$$

By noting that $\omega^2 x/g$ is the slope of the tangent to the parabola, Eq. (13) could have been written directly. When $\tan\alpha$, the slope of the tube, is less than $\omega^2 x/g$, the term $\left(\dfrac{\omega^2 x}{g} - \tan\alpha\right)$ is positive and the downward or inward pressure exceeds the pressure on the inner end of the prism.

It is now assumed that the prism used in developing Eqs. (12) and (13) is replaced by a new body of specific weight w_1 and total weight W_1. The forces acting on it are a pressure force parallel to the tube which is the same as $dp\,dA$ from Eq. (12), the reversed effective force $W_1\,\omega^2 x/g$ and the weight W_1. These forces have a

resultant R along the tube which will be taken as positive in the downward direction and which is

$$R = W\left(\frac{\omega^2 x}{g}\cos\alpha - \sin\alpha\right) - \frac{W_1\,\omega^2 x}{g}\cos\alpha + W_1 \sin\alpha$$

from which

$$R = (W - W_1)\left(\frac{\omega^2 x}{g}\cos\alpha - \sin\alpha\right) \tag{14}$$

or

$$R = (W - W_1)\left(\frac{\omega^2 x}{g} - \tan\alpha\right)\cos\alpha \tag{15}$$

With $W > W_1$ and $\tan\alpha < \omega^2 x/g$, R is positive and the light body will therefore travel down the tube. It will come to equilibrium again if it reaches a place where $\tan\alpha = \omega^2 x/g$, that is, a place where the tube and the tangent to the parabola are parallel. The force R is reversed if $W_1 > W$. If the tube is horizontal, an object lighter than the liquid always comes to the axis of rotation and a heavier object always moves away from the axis.[1]

It follows from the above that any homogeneous object placed in the rotating liquid of Fig. 52 will move to point O if it is lighter than the liquid and to the outside of the bottom if it is heavier. If its center of gravity is below the center of gravity of the displacement, a floating body will move to the level DF.

General Problems

92. The tank of water in Fig. 57 is given a uniform acceleration of 8.05 ft./sec.² toward the right. How deep is the water along AB? What is the pressure force on end AB if it is 18 in. wide?

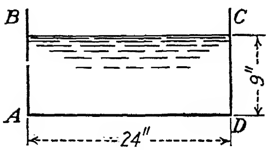


FIG. 57.

93. The tank of water in Fig. 57 is given a downward acceleration of 16.1 ft./sec.² What is the pressure in pounds per square foot on the bottom?

94. The cylindrical tank of Fig. 58 is half full of water and is rotated about its vertical axis. What speed of rotation will cause the water to reach the top? What will then be the maximum pressure in the tank? If the water is 3 ft. deep at the sides, what is the speed and how deep is it at the center?

[1] This analysis follows that by G. M. Minchin, "A Treatise on Hydrostatics," Vol. I, p. 91, Clarendon Press, Oxford, 1912.

95. The tank of Fig. 58 is half full of oil, specific gravity, 0.75. What speed of rotation is necessary to expose one-half of the bottom diameter? How much oil is lost in attaining this speed?

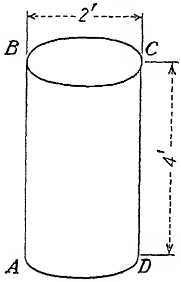


FIG. 58.

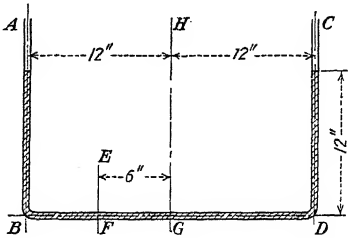


FIG. 59.—Open U-tube of uniform diameter containing mercury.

96. The U-tube of Fig. 59 is given a uniform acceleration of 4.02 ft./sec.^{2} toward the right. What is the depth in AB and the pressure at B, G and D?

97. The U-tube of Fig. 59 is rotated around HG so that the velocity of B is 10 ft. per sec. What is the pressure at B and G?

98. The U-tube of Fig. 59 is rotated about HG. At what angular velocity does the pressure at G become zero gage? What angular velocity is required to produce a cavity at G?

99. At what speed must the U-tube of Fig. 59 be rotated about AB to empty leg AB? What is then the pressure at G and D?

100. The U-tube of Fig. 59 is rotated about EF at such a speed that AB is empty. What is the angular velocity?

101. The tube of Fig. 60, containing mercury, rotates about AB. Compute the angular velocity (a) to make the pressure at A zero gage, (b) to form a cavity at A, (c) to empty AB.

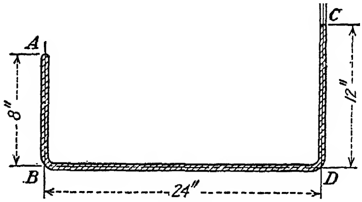


FIG. 60.

102. A tank of water is given an acceleration of 5 ft./sec.^{2} parallel to and down a plane which makes an angle of 30 deg. with the horizontal. What is the angular position of the surface? What is the pressure at a point 4 ft. below the surface?

103. The tank of Fig. 58 has a cover with a small hole at the center. If it is full of water and has an angular velocity of 8 rad. per sec., what is the pressure at B and A?

104. The tank of Fig. 58 contains 2 ft. of water covered by 1 ft. of oil (specific gravity, 0.75). What speed of rotation will cause the oil to reach B? What is then the pressure at A?

105. The tube of water in Fig. 61 is rotated about axis AB. What angular velocity is required to make the pressures at B and C equal? At that speed what and where is the minimum pressure in tube BC?

106. In Fig. 56 the distance $BC = 12$ in. and $\alpha = 45$ deg. What minimum angular velocity will cause a light particle in CD to move to tube E?

107. In Fig. 62 the lower vertical tube is 0.4 in. in diameter and contains mercury; the upper tube is 0.2 in. in diameter and contains oil (specific

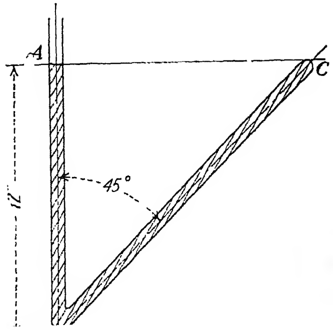


FIG. 61.

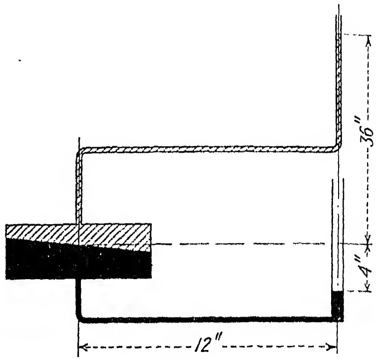


FIG. 62.

gravity, 0.8). The reservoir is large in area as compared with the tubes. What acceleration is required to produce the conditions shown in the figure?

CHAPTER V

DYNAMICS OF FLUIDS

38. Forces in Fluids in Motion.—In studying the flow of fluids there are in general five types of forces which must be considered. They are:

1. External forces such as those produced by gravity.
2. Forces caused by difference in pressure at various points.
3. Inertia forces represented by the product of mass and acceleration of fluid particles.
4. Forces due to viscosity, that is, shearing forces set up between adjacent particles of moving fluid.
5. Elastic forces due to compressibility of gases.

The first two of these types are involved in hydrostatics, the first four are present in flowing liquids, and all five types must be considered when dealing with flowing gas.

For the present, however, consideration will be given to a so-called ideal fluid which is assumed to be nonviscous and incompressible. The first assumption seems to be fairly justifiable for such fluids as water and air which, as has been pointed out in Chap. I, are relatively low in viscosity, while the hypothesis of an incompressible fluid appears to be reasonable for water but not for gases. However, the general procedure to be followed in studying the dynamics of fluids will be first to discuss the important characteristics of an ideal fluid and then to introduce the effects of viscosity and compressibility as qualifying factors.

39. Streamlines. Steady and Unsteady Motion.—In fluid motion, it is sometimes possible to draw lines through the fluid so situated that at any point the velocity vector is the tangent to the line which passes through that point, as shown in Fig. 63. These lines, which in general are curves in three dimensions, are known as streamlines.

When the motion is of such a character that the forces, velocity and pressure, are independent of the time, and for any given point do not change from one instant to the next, the fluid is said to be

in a state of steady motion. If at any one point these quantities are continually changing with time, the flow is called an unsteady motion. This distinction in the character of motions of fluids may be clarified by means of the concept of the streamline. In the case of steady motion, the streamlines remain unchanged in shape with time and may be considered as representing the paths followed by the fluid particles. If the flow is unsteady, then the streamlines may be considered only as curves drawn instantaneously through the fluid, and will be continually changing in shape from one instant to the next. In this case they do not represent the paths of the particles.

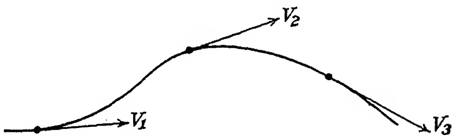


FIG. 63.—Definition of a streamline.

The fully developed flow of a current of air past an airplane wing or the flow of water from an opening in the bottom of a tank, the level in the tank being kept constant, may be considered as cases of steady motion. In the latter example, if the level of the water in the tank were allowed to vary during the discharge, the motion would be unsteady. The majority of engineering problems in fluid mechanics are concerned with flow of the steady type. While unsteady motion is sometimes of considerable importance, for the present the assumption of steady flow will be added to the other idealizing conditions.

40. Continuity of Flow.—If a state of steady motion exists in a mass of fluid in which there are no cavities, and if a small closed curve such as *mn* of Fig. 64 is considered within the fluid, then streamlines drawn through every point of this curve form what is known as a stream tube. Since the surface of this tube consists entirely of streamlines, there can be no flow of fluid across its boundary, and the tube thus has the appearance of a pipe with a solid wall, but, in general, with a variable cross section, as shown in Fig. 64. Furthermore, if there are no points at which fluid is either created or destroyed, then the mass of fluid passing any cross section of the tube must be a constant, that is, referring to Fig. 64,

$$\rho_a A_a V_a = \rho_b A_b V_b$$

or for any section

$$\rho A V = \text{constant} \tag{1}$$

where ρ represents the density of the fluid, A the area of the cross section and V the velocity at that section, while the subscripts a and b refer to any two particular sections of the tube. This equation represents the simplest general statement of the so-called condition of continuity which must be satisfied by most types of fluid motion. In the case of an incompressible fluid,

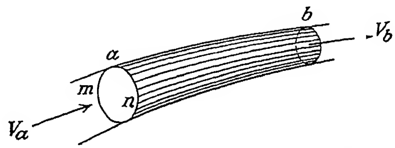


FIG. 64.—Flow through a stream tube.

there is no variation in the density ρ between any two points so the equation of continuity may then be written as

$$AV = \text{constant} \tag{2}$$

Strictly speaking, the cross-sectional area of a stream tube is an infinitesimal, but the condition of continuity may usually be applied to flows through passages of finite area. Thus in the case of a pipe, full of water flowing from one end to the other, its interior may be divided up into a large number of small stream tubes and the total quantity of fluid passing any cross section will be the sum of the quantities passing through the individual tubes at that section. If the areas of the individual tube cross sections are represented by $\Delta A_1, \Delta A_2, \ldots \Delta A_k, \ldots \Delta A_n$, and the corresponding velocities are $V_1, V_2, \ldots V_k, \ldots V_n$, then the total quantity of fluid passing this cross section of the pipe will be the limit of the sum of the quantities passing through the elementary stream tubes as their number is allowed to become infinite. From the definition of an integral

$$Q = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k V_k = \int V \, dA \tag{3}$$

If the velocity is uniform, or if V represents the average velocity and A the total cross section, then Eq. (3) may be written directly as

$$Q = AV \tag{4}$$

in which form it is known as the discharge equation. With A in square feet and V in feet per second, it gives the rate of flow or discharge through any cross section of a stream in cubic feet per second.

The mass flowing through any cross section is

$$Q_M = \rho A V \tag{5}$$

so that the constant of Eq. (1) is Q_M, the mass discharge. With A and V in foot and second units and ρ in slugs per cubic foot, Q_M is measured in slugs per second.

The application of Eqs. (4) and (5) to successive cross sections of a stream results, respectively, in

$$A_a V_a = A_b V_b = AV \tag{6}$$

and

$$\rho_a A_a V_a = \rho_b A_b V_b = \rho A V \tag{7}$$

These are the commonly used forms of the continuity equation, Eq. (7) being especially applicable to compressible fluids.

The condition of continuity is really a special case of the general physical law of the conservation of matter applied to the fluid within a stream tube.

41. Further Applications of Continuity.—The condition of continuity of flow as applied to the stream tubes of a fluid motion may be employed to advantage in studying the character of the flow around bodies immersed in a stream. This is particularly true if the motion is "two dimensional," that is, when the flow may be considered as being identical in pattern in any number of parallel plane cross sections. The motion of a stream of fluid past an indefinitely long cylinder in a direction perpendicular to the cylinder generators is an example of this type. In such a case the layers of fluid situated between adjacent streamlines, which are now plane curves in any one of the above-mentioned cross sections, may be considered as stream tubes. As an example, the streamlines for the flow of an unlimited mass of fluid around the cross section of an airplane wing or vane are shown in Fig. 65. The fluid is assumed to have a uniform velocity at a great distance upstream from the airfoil and the streamlines are chosen so that they are uniformly spaced in this region. The quantity of fluid passing between any two adjacent streamlines is therefore the same; furthermore, this quantity must, according to the condi-

tion of continuity, remain constant for the entire length of the stream tube. Thus in Fig. 65 it will be noted that, while at great distances ahead of and behind the airfoil the streamlines are uniformly spaced, they crowd together in the region immediately above the wing and spread apart below the wing. It may be concluded that at points above the wing the fluid has experienced an increase in velocity over that of the undisturbed stream, while below the wing the flow has been retarded.

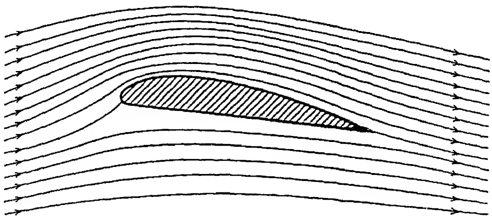
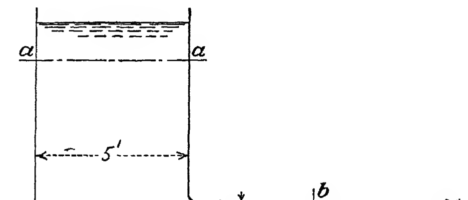


FIG. 65.—Flow past an airfoil section.

Example.—Water flows steadily at a rate of 225 g.p.m. from a cylindrical tank through the pipe and nozzle shown in the accompanying figure. Determine the average velocities at sections *aa* and *bb* and at the end of the nozzle.



Solution.—Since 1 gal. = 231 cu. in., the discharge is

$$Q = \frac{225 \times 231}{1728} = 30.1 \text{ cu. ft./min.} = 0.502 \text{ c.f.s.}$$

At section *aa*, $Q = A_a V_a$ and, since $A_a = \pi \times 25/4 = 19.6$ sq. ft.,

$$V_a = \frac{Q}{A_a} = \frac{0.502}{19.6} = 0.0256 \text{ ft./sec.}$$

At section *bb*

$$V_b = \frac{Q}{A_b} = \frac{0.502}{0.196} = 2.56 \text{ ft./sec.}$$

At the end of the nozzle

$$V_n = \frac{Q}{A_n} = \frac{0.502}{0.049} = 10.24 \text{ ft./sec.}$$

Problem 108. A circular pipe carrying water converges uniformly from an inlet diameter of 3.6 ft. to an outlet diameter of 1.2 ft. in a length of 100 ft. If the discharge is 748 g.p.m., determine the velocity at the inlet and at points 25 ft. apart along the axis of the pipe.

109. A circular stack has a diameter of 50 ft. at its base and converges uniformly to a diameter of 35 ft. at a height of 125 ft. Coal gas, having a specific weight of 0.025 lb. per cu. ft., enters the stack at the bottom. Its specific weight increases uniformly until at the top it has a value of 0.035 lb. per cu. ft. Calculate the velocity in feet per minute and the discharge in cubic feet per minute for every 25 ft. up the stack if the velocity at the top is 15 ft. per sec.

110. The streamlines of the flow past a long cylinder are uniformly spaced at distances of 2 in. apart far ahead of the cylinder, the velocity of the stream being 65 ft. per sec. Near the cylinder the streamlines crowd together so that the spacings between adjacent lines are 1 in., 1.25 in., and 1.50 in. What are the average velocities between these lines?

42. Energy of Fluids in Motion.—Fluids at rest or in motion must conform like all other matter to the law of conservation of energy. Energy, which is defined as the ability to do work, exists in several forms in nature. In the case of fluids, the kinetic, potential and pressure energies are of particular importance.

Kinetic energy is that energy which a mass possesses by virtue of its motion. A mass W/g having a velocity V has a kinetic energy given by the expression

$$\text{Kinetic energy} = \frac{WV^2}{2g} \tag{8}$$

In terms of the fundamental units of mass, length and time, kinetic energy has the dimension ML^2/T^2 and, in the English or foot-pound-second system, it is measured in foot-pounds.

Potential energy is that energy which a body possesses by virtue of its position above some horizontal datum plane. It represents the amount of work that the body is capable of doing if it is allowed to descend to the datum plane. This energy is equal in magnitude to the amount that must have been expended in raising the body to its assigned elevation. Thus for a body having a weight W and being located at a distance z above the datum plane mn of Fig. 66a, the potential energy is

$$\text{Potential energy} = Wz \tag{9}$$

As in the case of kinetic energy, this quantity is measured in foot-pounds.

Pressure maintained in a fluid represents a certain ability to do work. This may be shown by considering a closed pipe filled with liquid under a pressure p. If a tube is connected to the pipe, the surface in the tube stands at a height p/w, as shown in Fig. 66*b*. When a particle of liquid is removed from the top of the tube, the surface again rises to height p/w if the pressure p is maintained. In this manner any quantity could be lifted by the pressure to this

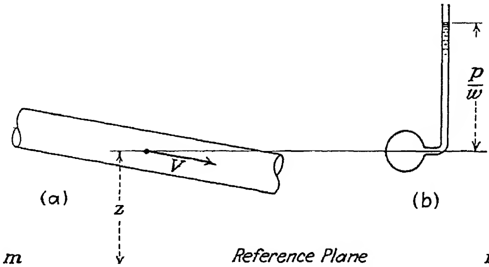


FIG. 66.—Potential energy and pressure energy.

height. In lifting a total weight W the work done by the pressure is

$$\text{Pressure energy} = W\frac{p}{w} \tag{10}$$

Since the expression Wp/w represents the energy content of a weight of fluid equal to W, the term p/w must therefore be equal to the energy per unit weight. In English units, it is measured in foot-pounds per pound, that is,

$$\frac{p}{w} \approx \frac{M}{LT^2} \div \frac{M}{L^2T^2} \approx L$$

which checks with the previous result that p/w represents a pressure head and therefore has the dimension of a length.

Referring to Eq. (8) which gives the kinetic energy of weight W of fluid, it is seen that the kinetic per unit weight is $V^2/2g$; since

$$\frac{V^2}{2g} \approx \frac{L^2}{T^2} \div \frac{L}{T^2} \approx L$$

the dimension is a length. The expression $V^2/2g$ is commonly called the velocity head and is usually measured in feet. It is

obvious from Eq. (9) that the potential energy per unit weight, relative to the datum, is z.

It is well to note that all the forms of energy discussed in connection with the motion of a fluid are relative, that is, each of the quantities represented by the expressions (8), (9) and (10) indicates the energy content of a certain weight of fluid referred to some arbitrary level or condition. Thus kinetic energy usually represents the difference in energy content between a particle of fluid in motion and one at rest. Potential energy is measured with reference to some arbitrarily chosen plane, so that it represents the difference in energy between a particle above the plane and one on it. In the case of pressure energy, the reference point is usually the zero on either the absolute or the gage pressure scale, depending on the nature of the problem at hand.

43. Bernoulli's Theorem.—With a knowledge of the various forms of energy that are present in a fluid in motion, the principle of conservation of energy may be applied to any flow of the steady type. This principle states that the total energy of any mechanical system must remain constant. For a particle of fluid of weight W located at a point where the pressure is p, the velocity is V, the elevation is z, the total energy is the sum of its kinetic, potential and pressure energies. Thus from Eqs. (8), (9) and (10)

$$\frac{WV^2}{2g} + \frac{Wp}{w} + Wz = \text{constant} \tag{11}$$

Canceling the term W, this becomes

$$\frac{V^2}{2g} + \frac{p}{w} + z = H \tag{12}$$

As explained in the last article, each of the terms on the left of Eq. (12) represents energy per pound of fluid and, having the dimension of a length, they are called velocity head, pressure head and elevation head, respectively. The sum H is called the total head.

If each term of Eq. (12) is multiplied by w, the weight per unit volume, then these terms represent energy per unit volume and the equation becomes

$$\frac{wV^2}{2g} + p + wz = wH$$

Substituting ρg for w, this equation takes the form

$$\frac{\rho V^2}{2} + p + \rho g z = \rho g H = E \tag{13}$$

in which E is the total energy per unit volume and is therefore a constant.

Equations (12) and (13) for the steady motion of an ideal fluid were first stated in 1738 by Daniel Bernoulli, a famous mathematician. They are known as Bernoulli's theorem and H and E are the Bernoulli constants. Whether Bernoulli's theorem is to be used in the form of Eq. (12) or of Eq. (13), it is essential that consistent units be employed throughout. Thus if the velocity is given in feet per second, the pressure should be expressed in pounds per square foot, the density in slugs per cubic foot and the elevation in feet.

Problem 111. A 6-in. water pipe carries 1200 g.p.m. at a pressure of 4 lb. per sq. in. What is the Bernoulli constant relative to a datum 10 ft. below the pipe?

112. A stream of water 8 ft. deep flows with a uniform velocity of 12 ft. per sec. If elevations are measured from the bottom of the stream, what are the values of the potential, pressure and kinetic energy per pound and the Bernoulli constant for points on the surface, at mid-depth and at the bottom of the stream?

113. A natural wind current has a uniform velocity of 22 m.p.h. Assuming the air to have a constant specific weight of 0.0765 lb. per cu. ft., compute the pressure, potential and kinetic energies per unit volume and the Bernoulli constant for altitudes of zero and 2000 ft.

114. A horizontal water pipe reduces gradually from a diameter of 18 in. at point A to 6 in. at point B. The flow in the line is 5 c.f.s. and the pressure at B is 10 lb. per sq. in. What is the pressure at A?

115. A pipe line enlarges from a diameter of 6 in. at A to 12 in. at point B, which point is 6 ft. higher than A. The flow of water in the line is 6 c.f.s. and the pressure at A is 10 lb. per sq. in. What is the pressure at B?

116. In Prob. 109, what is the pressure at the base of the stack?

44. Alternative Proof of Bernoulli's Theorem.—It should be mentioned that Bernoulli's theorem is true for all points in a body of fluid in motion provided the streamlines originate in a region where the fluid is at rest or where all particles are moving with the same total energy. Under such conditions the total energy of the particles moving along any two streamlines is the same and the constants H and E have the same values for all points in the fluid. If, however, the motion is nonuniform in character at its

beginning so that the particles which move along different streamlines begin their journeys with different total energy contents, then Bernoulli's theorem cannot be applied to points that are not on the same streamline.

This restriction on the application of Bernoulli's theorem may be shown more clearly by basing its derivation on a consideration of the forces which act on an infinitesimal element of fluid moving along a streamline. Let the length of such an element be dl measured along the streamline, its cross-sectional area dA and the velocity with which it is moving V, as shown in Fig. 67. On the left-hand end of the element there is a pressure p, and on the other end a larger pressure $p + dp$, where dp represents the change in pressure that occurs in passing from one end of the element to the

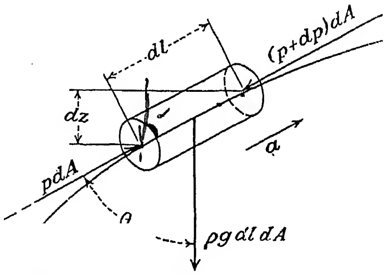


FIG. 67.—Equilibrium of a fluid element.

other. Thus the net force on the element produced by the pressures is

$$(p + dp)dA - p\,dA = dp\,dA$$

Considering an upward displacement to be positive this force, being downward in direction, must appear in force equations as $-dp\,dA$.

This same convention in regard to the sign of forces will be employed in the remainder of this development. The acceleration is assumed to be upward, that is, positive in sign.

There is also a component of the weight of the element which acts in the axial direction. From Fig. 67 it is evident that this component is

$$-\rho g\,dl\,dA \cos\theta$$

where θ is the angle between the vertical and the tangent to the

streamline at the point where the element is located. But from the figure

$$\cos \theta = \frac{dz}{dl}$$

dz being the projection of dl on the vertical. The desired component of the gravitational force is then

$$-\rho g\, dl\, dA \frac{dz}{dl} = -\rho g\, dA\, dz$$

If the velocity V is considered as varying from point to point along the streamline, that is, V is a function of l, then the acceleration is the rate of change of the velocity with time, so that

$$a = \frac{dV}{dt} = \frac{dV}{dl}\frac{dl}{dt}$$

But dl/dt is by definition the velocity, so that

$$a = V\frac{dV}{dl} = \frac{1}{2}\frac{d(V^2)}{dl}$$

The effective force required to accelerate a mass is the product of the mass and the acceleration so that the reversed effective force is

$$-\rho\, dl\, dA \cdot \frac{1}{2}\frac{d(V^2)}{dl} = -\frac{\rho}{2}\, dA\, d(V^2)$$

Now, according to D'Alembert's principle, the sum of the applied forces and the reversed effective force must be zero, that is,

$$-dp\, dA - \rho g\, dA\, dz - \frac{\rho}{2}\, dA\, d(V^2) = 0$$

On canceling the area dA and changing signs, this expression becomes

$$dp + \rho g\, dz + \frac{\rho}{2}\, d(V^2) = 0 \qquad (14)$$

which is really a differential equation relating the pressure, elevation and velocity of the fluid element. The differential quantities in the above equation represent the changes in the various terms which occur in a distance dl measured along

a streamline. The equation may therefore be integrated along such a line, giving as the result

$$p + \rho gz + \frac{\rho V^2}{2} = \text{constant} \tag{15}$$

Since the left side is identical with that of Eq. (13), the constant of integration may be put equal to E. In general, then, E will have different values depending on which streamline is chosen for the integration, except when all the streamlines originate under the same conditions, in which case E is constant for all points in the fluid.

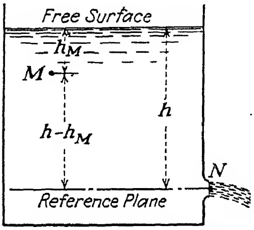


FIG. 68.—Velocity of efflux.

45. Torricelli's Theorem.—Bernoulli's equation is applied readily to the proof of a theorem that was demonstrated by Torricelli in 1644, nearly a century before Bernoulli's work.

The tank of Fig. 68 is filled with liquid to a height h above the center of the opening N in its side. It is assumed that the tank is large in comparison with the opening, that the free surface and the jet are acted upon by the same atmospheric pressure and that the former is maintained at a constant level. The pressure head at M is equal to $h_M + \frac{p_a}{w}$ on the absolute scale, and the potential head, taking a reference plane through N, is $h - h_M$. Having assumed that the tank is large in proportion to the opening, the velocity head at M may be neglected. Then the Bernoulli constant H for point M is

$$H = \left(h_M + \frac{p_a}{w}\right) + (h - h_M) = \frac{p_a}{w} + h$$

and the value of H for point N is

$$H = \frac{p_a}{w} + \frac{V^2}{2g}$$

in which V is the velocity at N. Equating these values of H, it is found that

$$V = \sqrt{2gh} \tag{16}$$

This will be recognized as the velocity attained by a body in falling freely in a vacuum through a height h. The fact that the velocity of efflux and the velocity attained in free fall are equal is known as Torricelli's theorem.

46. The Siphon.—The action of a siphon may be studied advantageously by the use of Bernoulli's theorem. If the tube shown in Fig. 69 has one end submerged in a tank of liquid, the end D being below the free surface, and the tube is filled with liquid, there will be a discharge at D, the lower end of the tube. The free surface and the discharging jet are subject to the same atmospheric pressure p_a. The pressure head at B is then p_a/w, the elevation head referred to a plane through D is h and, assuming the tank to be large in comparison with the tube, the velocity head at B may be neglected. At D the velocity head is $V^2/2g$, V being the velocity of efflux, and the pressure head is p_a/w. Equating the expressions for the total head at these two points gives the result

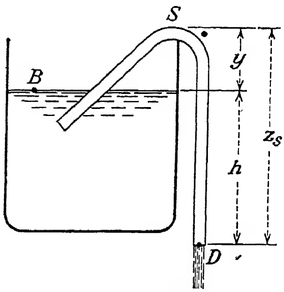


FIG. 69.—The siphon.

$$\frac{p_a}{w} + h = \frac{p_a}{w} + \frac{V^2}{2g}$$

or

$$V = \sqrt{2gh} \tag{17}$$

If the area at the outlet end of the siphon is A, then the discharge is

$$Q = AV = A\sqrt{2gh}$$

In order for the tube to flow full of liquid, the absolute pressure at the summit of the siphon must be greater than zero. The velocity at that point may be determined by applying the equation of continuity in the form

$$A_sV_s = AV$$

where A_s and V_s are the area and velocity, respectively, at the summit. The pressure may then be calculated by means of Bernoulli's theorem, the total head, determined from the condi-

tions at either the inlet or the outlet, being equated to the total head at the summit. If the result of such calculations yields a negative absolute pressure at the summit, then the application of the equation of continuity is not justified, for the siphon will flow only partly full in the lower part of the outlet leg. For this reason the maximum elevation z_s of the crest of a siphon flowing full is p_a/w if the diameter is uniform. In actual operation this distance is further limited by separation of air and vapor from the liquid at low pressure.

Problem 117. The siphon of Fig. 69 has a uniform diameter of 6 in., h is 9 ft. and z_s is 15 ft. Find the velocity and quantity of water being discharged. What is the pressure at the summit?

118. In Fig. 69 the opening at D is 3 in. in diameter, the pipe is 6 in. in diameter, h is 9 ft. and z_s is 15 ft. Find the velocity at D, discharge, and pressure at the summit. The liquid is water.

119. In Fig. 69 the pipe is 6 in. in diameter, z_s is 40 ft., y is 9 ft. and the liquid is water. Compute the maximum discharge of the siphon. What are the pressure at s and the head under which the siphon is operating?

120. Work Prob. 118, changing the liquid to oil with specific gravity of 0.75.

47. Measurement of Velocity and Pressure.—The determination of the velocity and pressure at points in a moving fluid

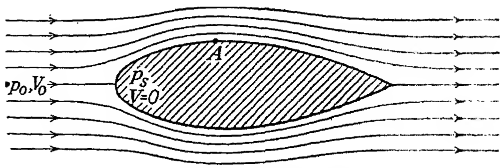


FIG. 70.—Flow past a body.

presents a group of problems to which Bernoulli's theorem may be applied with considerable success. If it is required to find the velocities and pressures in a body of initially stationary fluid caused by the motion of a solid object through the fluid, it is convenient to consider the motion of the fluid relative to the object rather than relative to fixed axes. The body may be considered as stationary with a stream of fluid moving past it such that at a great distance from the body the stream behaves as a uniform current with a velocity equal but opposite to that originally possessed by the body. Such a flow is illustrated in Fig. 70.

In general, there is at least one point, usually somewhere on the forward portion of the body, where the fluid is brought completely to rest, such a point being known as a stagnation point. If the pressure and velocity in the undisturbed part of the stream are known and are represented by p_0 and V_0, respectively, then the pressure at the stagnation point may be calculated by applying Bernoulli's theorem to this point and the one in the undisturbed fluid. In order to simplify the problem, it will be assumed that these two points are both at the same distance above the reference plane so that their elevation heads are equal. In the case of gases such as air, this assumption is justified because of the low specific weight of the fluid, except in cases where extremely large differences in elevation are considered, as, for example, in meteorology, in which the behavior of the earth's atmosphere is studied.

The application of Bernoulli's theorem in the form of Eq. (13) gives the following relationship:

$$p_0 + \frac{\rho V_0^2}{2} = p_s \tag{18}$$

where p_s represents the pressure at the stagnation point. Writing this equation in the form

$$p_s - p_0 = \frac{\rho V_0^2}{2} \tag{19}$$

it may be seen that the rise in pressure produced by bringing a fluid to rest at a stagnation point is equal to $\rho V_0^2/2$. This last expression is commonly known as the impact or dynamic pressure of the stream and in aeronautics is usually denoted by the symbol q.

If the position of the stagnation point on a body is known, the measurement of the pressures at that point and in the undisturbed fluid makes it possible to determine the velocity of the fluid stream, for Eq. (19), when solved for V_0, gives

$$V_0 = \sqrt{\frac{2(p_s - p_0)}{\rho}} \tag{20}$$

The stagnation-point pressure may be determined experimentally by inserting a small tube into the interior of the body so that one

end of it is brought out flush with the surface at the stagnation point. The other end is connected to a suitable manometer or draft gage, from the reading of which p_s may be determined.

The same method may also be employed for measuring the pressure distribution over the surfaces of such objects as airship hulls, tall buildings and stacks, and many other bodies which are located in moving fluid. In such problems it is necessary to insert a large number of pressure tubes at various points on the

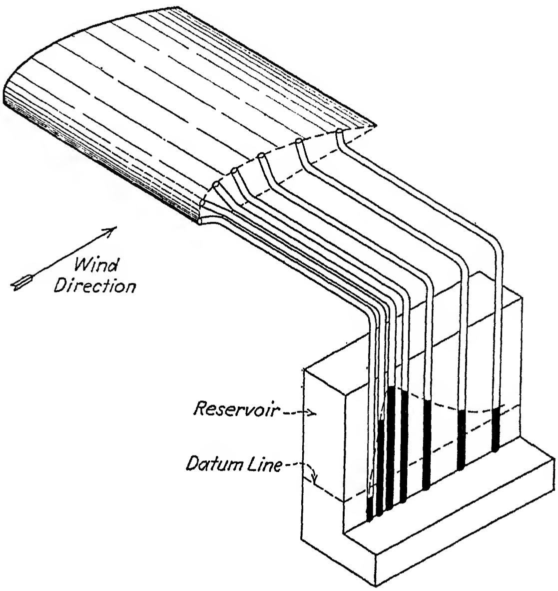


FIG. 71.—Measurement of pressure distribution over upper surface of an airfoil.

surface of the body, each tube being connected with an individual manometer. An arrangement of this kind for the determination of the pressure distribution on an airplane wing is shown in Fig. 71.

In certain cases it is of importance to know not only the pressure distribution but also the variation of velocity over the surface of a body placed in a stream of fluid. If, for example, the pressure at a point such as A in Fig. 70 has been determined by the experimental method described above and is designated by p, then the corresponding velocity V may readily be found by writing Bernoulli's equation for the point A and for a point in

the undisturbed stream. Again assuming that the difference in potential energy between these two points is negligible, then

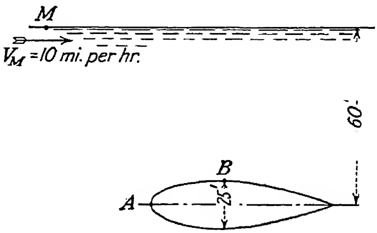
$$p_0 + \frac{\rho V_0^2}{2} = p + \frac{\rho V^2}{2}$$

from which the velocity at A is

$$V = \sqrt{V_0^2 + \frac{2}{\rho}(p_0 - p)} \tag{21}$$

It may be shown from this relationship that the maximum pressure which may exist in a fluid stream is that which occurs at a stagnation point where $V = 0$.

Example.—A submarine travels forward at a velocity of 10 m.p.h. in salt water (specific weight, 64.2 lb. per cu. ft.) with its axis horizontal and at a depth of 60 ft. (*a*) What is the gage pressure at the nose? (*b*) If its largest cross section is 25 ft. in diameter and the pressure at the highest point on this cross section is 15 lb. per sq. in. gage, what is the relative velocity at that point?



Solution.—*a*. Considering the relative motion of a stream of water past the stationary submarine, the flow is steady and Bernoulli's theorem may be applied. Point A in the figure is a stagnation point; writing Bernoulli's theorem between A and point M on the surface and considering the surface to be at datum, the result is

$$\frac{p_A}{w} + z_A = \frac{V_M^2}{2g}$$

The atmospheric pressure on the water surface is taken as zero. Then

$$p_A = w\left(\frac{V_M^2}{2g} - z_A\right)$$

and since $z_A = -60$ and 60 m.p.h $= 88$ ft./sec.

$$\begin{aligned} p_A &= 64.2\left[\frac{(10 \times 88/60)^2}{2 \times 32.2} + 60\right] \\ &= 64.2(3.34 + 60) = 4066 \text{ lb./sq. ft. gage} \\ &= 28.2 \text{ lb./sq. in. gage} \end{aligned}$$

b. To find the velocity at point B, Bernoulli's theorem may be applied either to points M and B or to A and B. Considering M and B, the result is

$$\frac{p_B}{w} + z_B + \frac{V_B^2}{2g} = \frac{V_M^2}{2g}$$

from which

$$V_B = \sqrt{2g\left(\frac{V_M^2}{2g} - \frac{p_B}{w} - z_B\right)}$$

Putting $\frac{p_B}{w} = 15 \times \frac{144}{64.2} = 33.6$ ft. and $z_B = -(60 - 12.5) = -47.5$ ft., then

$$\begin{aligned} V_B &= \sqrt{64.4(3.34 - 33.6 + 47.5)} \\ &= \sqrt{1110} = 33.3 \text{ ft./sec.} \\ &= 22.8 \text{ m.p.h.} \end{aligned}$$

Problem 121. What is the dynamic pressure head at a stagnation point on a body immersed in a stream of water moving with a velocity of 20 ft. per sec.? What is the total pressure head at this point if it is 10 ft. below the water surface?

122. The gage pressure head at a stagnation point in a stream of standard air is 3 in. of water. What is the velocity of the stream?

123. The velocity and pressure of an airstream are 60 m.p.h. and 14.7 lb. per sq. in. abs. and, at a point on a body immersed in it, the pressure is 14.27 lb. per sq. in. abs. What is the local velocity at this point? Density of air is 0.00225 slug per cu. ft. and may be assumed constant.

48. Measurement of Static Pressure.—The determination of the static pressure of a fluid stream at points that are not on the surface of a body or at the walls of a conduit requires special devices. In the case of fluid flowing through a pipe, the pressure at the wall may be measured by inserting a piezometer tube. The end of the tube must be flush with the inside wall of the pipe and any burrs or projections around the hole in the tube must be eliminated. With such an arrangement, the flow is entirely parallel to the pipe wall so that, for a pipe of a given size, the pressure is unaffected by changes in velocity. For this reason the pressure transmitted through the piezometer tube is known as the static pressure.

When it is desired to know the static pressure at points in the fluid which are not on the wall, some means must be provided for measuring that pressure without introducing the influence of the velocity. This might be done by moving the piezometer tube and the pipe wall in which it is placed out into the stream without changing the character of the flow.

In practice this may be accomplished by the use of an instrument such as Ser's disk, shown in Fig. 72*a*. This device consists of a small tube inserted into a flat circular plate having a diameter

usually four or five times that of the tube. The end of the tube is open and is just flush with the surface of the disk, the edges of which are usually beveled on the lower surface. In use the instrument is placed in a stream so that the surface of the plate is parallel to the direction of flow. No appreciable disturbance is produced by the instrument in the immediate neighborhood of the hole in the disk. The fluid flows smoothly over the upper surface of the disk without change in its original velocity and only the static pressure is transmitted through the hole into the tube, whence it may be led to a manometer for measurement.

The static-pressure tube shown in Fig. 72*b* is another device for the same purpose. It consists of a tube which is bent to form two legs at right angles to each other. The end of the

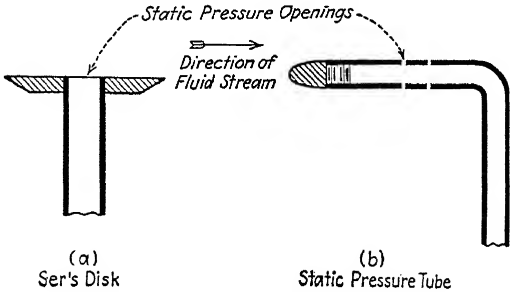


FIG. 72.—Static-pressure instruments.

horizontal leg is fitted with a rounded or tapered nose so as to minimize the disturbance caused by the tube. At some distance back from the nose, usually six or seven times the diameter of the tube, a number of very small holes are drilled through the wall. The vertical leg of the instrument serves as a support by means of which the device may be held in a fluid stream with the horizontal leg parallel to the direction of the flow. The fluid then moves smoothly past the small holes and the static pressure is transmitted through them into the vertical leg which is connected to the manometer. In order to eliminate any interference caused by the vertical leg of the tube, it is necessary that the bend be placed some distance downstream from the small holes.

Both Ser's disk and the static-pressure tube must be very carefully aligned with the direction of the stream at the point where the pressure is to be measured. There are several types of directionometers which are capable of measuring simultane-

ously without special alignment the static pressure, velocity and direction of a stream, the principle upon which they operate being based on Bernoulli's theorem. The description and theory of these devices, however, are beyond the scope of this book.[1]

49. The Pitot Tube.—The measurement of the velocity of a fluid stream may be made by utilizing the effect of the dynamic pressure. The most frequently employed instrument for this purpose is the so-called Pitot tube, named after the French scientist who was the first to use it. A drawing of a typical modern Pitot tube for use in air is shown in Fig. 73. The end

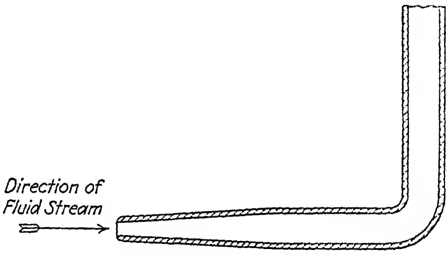


FIG. 73.—The Pitot tube.

of the tube pointing upstream is left open while the other end is connected to a manometer. Hence fluid cannot flow through the tube and the nose is to be regarded as a stagnation point. The theory of the first portion of Art. 47 leading to the development of Eq. (20) may thus be applied directly so that, if the total pressure at the nose of the instrument is p_s and the static pressure is p_0, then the velocity of the flow is

$$= \sqrt{2(p_s - p_0)} \qquad (22)$$

When the measurement of velocity is the chief concern, the Pitot tube just described may conveniently be combined with a static tube to form the Pitot-static tube, an example of which is shown in Fig. 74. In this instrument, the wall of the Pitot tube portion is constructed in the form of an annular passage, in the outer surface of which the static-pressure holes are drilled. The

[1] OWER, E., "The Measurement of Air Flow," Chapman and Hall, Ltd., London, 1927.

total pressure of the stream is thus carried through the central passage while the static pressure is transmitted through the outer passage. These two passages may be connected to opposite ends of a differential manometer, which then indicates the head equivalent to the dynamic pressure.

In this type of instrument, interference and misalignment produce errors. The errors in static and total pressures tend to compensate so that the instrument is much less sensitive to misalignment than either the simple static or the Pitot tubes. The National Physical Laboratory in England has developed an instrument of this kind for use in airstreams which may be turned

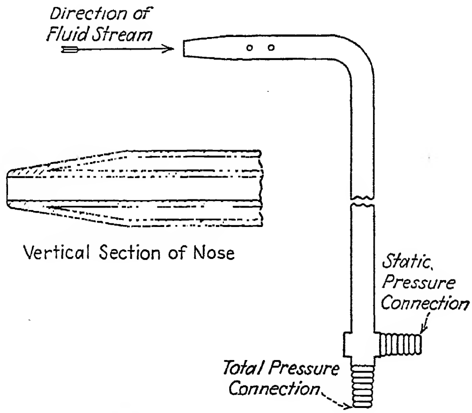


Fig. 74.—Pitot-static tube for air.

as much as 25 deg. from the direction of flow with a resultant error of only 2.3 per cent in the velocity.

Another type of Pitot-static tube, in which the total pressure and static-pressure tubes are separated, is shown in Fig. 75. Instruments of this type are frequently used as air-speed indicators for airplanes.

Pitot-static tubes are often used for measurements in water and some are designed for permanent installation in a pipe line. Figure 76 shows the head and part of the shank of a Pitot tube used by Professor Gardner S. Williams in his researches on flow of water in pipes. The ellipsoidal head is about three-quarters of an inch in length. The opening at the nose is at the stagnation point. By means of tubes passing through the shank, this opening is connected to one leg of a differential manometer and

all the pressure openings on the side are connected to the opposite leg. The manometer scale or recording instrument may be

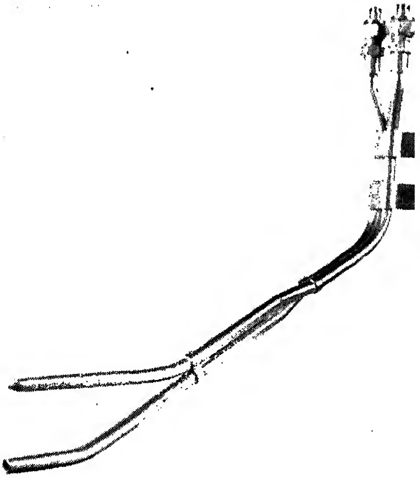


FIG. 75.—Pitot and static tubes of an air-speed indicator. (*Courtesy Pioneer Instrument Company.*)

arranged to read or record pressure difference, head, velocity or rate of flow. In making velocity measurement Eq. (22) may be applied. However, in the case of liquids, such as water, it is convenient to measure the pressure difference in terms of the equivalent head of the flowing liquid and to replace w/ρ by g. The expression for velocity then becomes

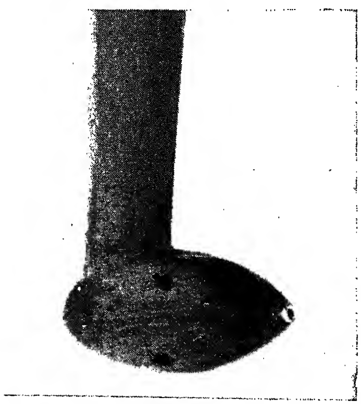


FIG. 76.—Pitot-static tube for use in water. (*Courtesy L. E. Ayres, Mem. A.S.C.E.*)

$$V = \sqrt{2g(h_s - h_0)} \qquad (23)$$

All the equations of this article must be modified by coefficients determined by calibration of each particular instrument. Such calibrations are required principally because of interference effects produced by the presence of the tube in the stream. In practice, Eqs. (22) and (23) take the forms

$$V = C\sqrt{\frac{2(p_s - p_0)}{\rho}} \tag{24}$$

$$V = C\sqrt{2g(h_s - h_0)} \tag{25}$$

in which the quantity C is a coefficient determined by calibration.

In the case of a Pitot-static tube intended to measure discharge, the calibration also corrects for the effect of nonuniform velocity distribution.

Problem 124. A Pitot-static tube placed in an airstream is connected to a manometer containing alcohol (specific gravity, 0.80). If the manometer reading is 6.5 cm. of alcohol, what is the speed of the stream?

125. A Ser's disk and a Pitot tube are placed at a depth of 5 ft. in a stream of water having a velocity of 15 ft. per sec. What heads will be indicated by these instruments?

126. A Pitot-static tube mounted on an airplane is attached to a pressure-type gage which indicates velocity in miles per hour. If the gage is calibrated for standard air, what is the true air speed when the indicated speed is 175 m.p.h. and the density is 0.00212 slug per cu. ft.?

50. The Venturi Meter.—The Venturi meter is a device which is widely used for measuring the quantity flowing in pipe lines.

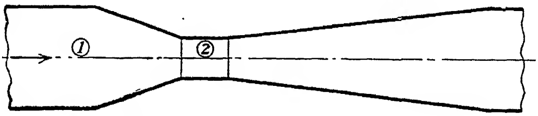


FIG. 77.—The Venturi meter.

It was invented in 1881 by Clemens Herschel, an American engineer, and was named by him in honor of Venturi, an early Italian experimenter in hydraulics.

This device consists of a tube by means of which the pipe is reduced and again enlarged, usually to its original size, and generally placed in a horizontal position as shown in Fig. 77. It has an inlet section which is tapered or rounded, a short section of constant diameter, known as the throat, and a gradually diverging section or outlet having an angle between the axis and the wall of 3 or 4 deg., never in excess of 6 deg. The area, pressure and velocity at the inlet are A_1, p_1 and V_1, respectively, and the corresponding quantities at the throat are A_2, p_2 and V_2. Then equating the values of the total head at these points as given by Bernoulli's equation,

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} \tag{26}$$

The equation of continuity is

$$A_1V_1 = A_2V_2$$

from which

$$V_2 = \frac{A_1V_1}{A_2} = nV_1$$

where $n = A_1/A_2$. If this value of V_2 is substituted in Eq. (26), it is found that the velocity at the inlet is

$$V_1 = \sqrt{2g\frac{1}{(n^2 - 1)}\left(\frac{p_1}{w} - \frac{p_2}{w}\right)} \tag{27}$$

or letting $K = \sqrt{\dfrac{1}{(n^2 - 1)}}$

$$V_1 = K\sqrt{2g\left(\frac{p_1}{w} - \frac{p_2}{w}\right)} \tag{28}$$

The discharge or quantity of fluid flowing through the meter is

$$Q = A_1V_1 = A_1K\sqrt{2g\left(\frac{p_1}{w} - \frac{p_2}{w}\right)} \tag{29}$$

From Eqs. (28) and (29) it appears that either the velocity or discharge through a Venturi tube can be computed after measuring the difference between pressures at the inlet and throat. This measurement may be made by providing openings in the wall at the entrance and throat. The pressures are transmitted to gages or recording devices which may be graduated to give discharge or velocity directly. Figure 78 shows a Venturi meter for use in a water line.

For accurate results the discharge from Eq. (29) must be modified by a coefficient determined by experiments. This coefficient corrects for the effect of loss of energy and nonuniform distribution of velocity, which was assumed to be uniform in the development of the equations. The values of the coefficient range from 0.95 to slightly more than unity.

51. The Venturi Meter for Gases.—The theory of the Venturi meter as developed in Art. 50 is applicable to the flow of gases

as well as liquids if conditions are such that the fluid may be considered as incompressible. The velocity or quantity of a gas flowing through a pipe line may then be determined by means of Eqs. (28) and (29). In dealing with air flow, the density

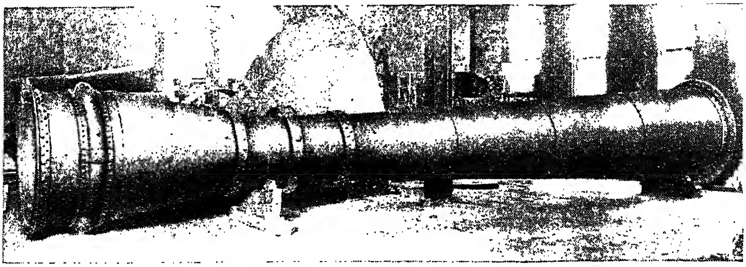


FIG. 78.—Venturi meter for use in a water line. (*Courtesy Builders Iron Foundry, Providence, R.I.*)

of the fluid rather than the specific weight is usually employed. If w/g is replaced by ρ, Eqs. (28) and (29) become

$$v_1 = K\sqrt{\frac{2(p_1 - p_2)}{}} \tag{30}$$

and

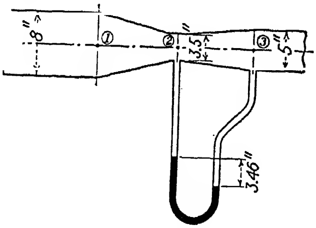
$$Q = A_1K\sqrt{\frac{2(p_1 - p_2)}{}} \tag{31}$$

A comparison of Eq. (30) with Eq. (22) for the Pitot tube shows that the relations between velocity and pressure difference are identical except for the presence of the factor K in the Venturi-meter formula. Recalling that $K = \sqrt{\frac{1}{n^2 - 1}}$, it is seen that K may be given any desired value by varying n, the ratio of inlet and throat areas. For $n = \sqrt{2}$, $K = 1$, while for $n > \sqrt{2}$, $K < 1$, so that for a given velocity the pressure difference must be larger than that obtained with a Pitot-static tube. This fact is a decided advantage in the measurement of gas flow, because with the Pitot-static tube the pressure difference is likely to be quite small. The use of a Venturi meter with a proper value of n appreciably magnifies the pressure difference, so that more accurate readings may be obtained.

All the devices used for measurement of velocity and discharge of gases require calibration just as in the case of the

instruments used with liquids. Additional errors are introduced when the fluid is compressible, but for low velocities these are usually negligible. A detailed discussion of the effects of compressibility will be given in Chap. XIII.

Example.—In a horizontal pipe line water flows from an 8-in. pipe into a 5-in. pipe, between which is placed a Venturi tube having a 3.5-in. throat. A U-tube connected to the throat and outlet sections of the Venturi as shown in the figure indicates a pressure difference of 3.46 in. of mercury. Determine the velocity at the Venturi throat and the discharge.



Solution.—Applying Bernoulli's theorem between points 2 and 3, the equation obtained is

$$\frac{p_2}{w} + \frac{V_2^2}{2g} = \frac{p_3}{w} + \frac{V_3^2}{2g}$$

But from continuity

$$A_2V_2 = A_3V_3$$

and

$$d_2^2V_2 = d_3^2V_3$$

$$V_3 = V_2\left(\frac{d_2}{d_3}\right)^2 = V_2\left(\frac{3.5}{5}\right)^2 = 0.49V_2$$

Then substituting this value of V_3 in the first equation,

$$\frac{V_2^2}{2g}(1 - 0.24) = \frac{p_3}{w} - \frac{p_2}{w}$$

or

$$V_2 = \sqrt{\frac{2g\left(\frac{p_3}{w} - \frac{p_2}{w}\right)}{0.76}}$$

Now

$$\frac{p_3}{w} - \frac{p_2}{w} = \frac{3.46}{12} \times 13.6 - \frac{3.46}{12} \times 1$$
$$= \frac{3.46}{12} \times 12.6 = 3.63 \text{ ft. of water}$$

The velocity at the throat is

$$V_2 = \sqrt{\frac{64.4 \times 3.63}{0.76}} = \sqrt{308} = 17.5 \text{ ft./sec.}$$

The discharge is

$$= A_2V_2 = \pi\left(\frac{3.5}{12}\right)^2 \times 17.5$$

$$= 1.17 \text{ c.f.s.}$$

Problem 127. A Venturi meter in a 12-in. water line has a 6-in. diameter throat. When the flow is 4 c.f.s. and the pressure at the throat is 20 lb. per sq. in. gage, what is the pressure at the upstream end of the meter?

128. A Venturi meter with a 6-in.-diameter throat placed in a horizontal water line 18 in. in diameter has connections from the inlet and throat to a U-tube containing mercury. What are the velocity and discharge through the line when the difference in level of the mercury is 12 in.?

129. A Venturi meter having a 6-in.-diameter throat is installed in a 12-in. pipe line carrying gas (specific weight, 0.047 lb. per cu. ft.). The pressure taps at the throat and inlet are connected to opposite sides of a simple U-tube containing water. What is the difference in level of the water columns when the velocity at the throat is 17.5 ft. per sec.?

52. Energy Losses in Fluids.—Whenever a real fluid is in motion, a certain portion of its total energy content is consumed in overcoming resistance to the flow. This situation is analogous to the operation of any machine in that the energy output is always less than the input. Bernoulli's constant at any one point in a flowing fluid is

$$H = \frac{p}{w} + \frac{V^2}{2g} + z$$

but the statement that the total energy and therefore the total head H is constant along a streamline requires some qualification. At point 1 in a flow, the total head is

$$H_1 = \frac{p_1}{w} + \frac{V_1^2}{2g} + z_1$$

while, for a second point 2 on the same streamline,

$$H_2 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

In passing along the streamline from point 1 to 2, the fluid loses a portion of its original energy content. The value of H_2 is therefore less than H_1 by h_l, the latter quantity representing the lost energy per unit of weight, that is, lost head. Then

$$H_1 = H_2 + h_l$$

or

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 + h_l$$

In this equation all the energy of the fluid is accounted for and this relationship is therefore a correct statement of the principle of conservation of energy applied to fluids in motion.

The nature of the loss in head in a flow is frequently quite complicated. In general, it is due to the effect of viscosity and to the manner in which the fluid flows past solid boundaries. For example, in the case of flow through a pipe, losses exist because of the viscosity of the fluid and because energy is required to make the fluid flow through valves or bends or past obstructions. The loss of energy consists in the transformation of part of the original energy content of the fluid into heat, and in most cases this heat cannot be reconverted into pressure, kinetic or potential energy. The detailed study of the loss in head in fluid flows will be taken up in later chapters.

Problem 130. A horizontal water line reduces from a diameter of 18 in. to 12 in. The pressure at the downstream end of the reducer is 22 lb. per sq. in. and the flow is 10 c.f.s. What is the pressure at the upstream end if the head lost in the reducer is 3 ft.?

131. A water line changes from a diameter of 6 in. at A to a diameter of 12 in. at B, which is 4 ft. above A. The pressures at A and B are 10 lb. per sq. in. and 9 lb. per sq. in., respectively, and the velocity at B is 5 ft. per sec. Find the lost head and the direction of flow.

132. The loss in head between the inlet and throat of a Venturi meter placed in a water main is 2 ft. What is the velocity in the pipe when the pressure difference is 4 lb. per sq. in. and the ratio of inlet to throat diameter is three?

53. Cavitation.—There are a number of important cases of fluid motion in which cavities or holes are formed in the interior of the fluid. This phenomenon is described by the term cavitation. The analytical methods of the previous articles cannot be applied to flows in which cavitation has occurred, but they are useful, nevertheless, in rigorously defining the conditions at the beginning of cavitation and in indicating the nature of the phenomenon.

A simple example of cavitation is found in static liquids. If a cylinder full of water is fitted with a piston which is initially in contact with the water and which is then gradually moved

away from the fluid, a space is formed between the piston and the water in which no fluid is present. However, as the pressure in this region is reduced, the water will reach its boiling point at a temperature below the normal value, so that actually the cavity will be filled with water vapor. A similar situation may arise when the fluid is in motion. If, for example, the piston of a reciprocating pump is given an acceleration so high that the water cannot maintain contact with the piston on the suction stroke, a cavity is formed and a disruption of the normally continuous flow takes place.

Cavitation is also found in fluid motions in which the velocity at some point reaches a value that is sufficiently high to cause

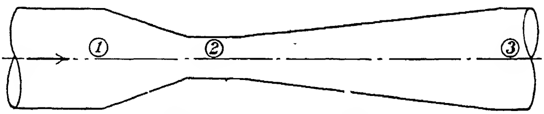


FIG. 79.—Venturi tube.

the pressure to approach zero absolute. A simple case of this kind is found in the flow of liquid through a Venturi tube.

Let it be supposed that the Venturi tube of Fig. 79 is a portion of a pipe line carrying water and assume for the present that the flow through the pipe is continuous. If a datum plane through the horizontal axis of the pipe is selected, thereby eliminating elevation head, the relation between pressures and velocities at the inlet and throat is

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} \tag{32}$$

Solving for p_2/w and substituting $V_2 = nV_1$, the result is

$$\frac{p_2}{w} = \frac{p_1}{w} - (n^2 - 1)\frac{V_1^2}{2g} \tag{33}$$

An inspection of this equation shows that, as V_1 is increased, p_2/w becomes smaller. It is clear that a condition may be reached in which the velocity becomes sufficiently high so that the absolute pressure p_2 approaches zero. The velocity at the inlet at which this condition is attained may be determined by letting $p_2 = 0$ in Eq. (33). The result is

$$V_1 = \sqrt{\frac{2g}{n^2 - 1} \frac{p_1}{w}} \tag{34}$$

A zero value of the absolute pressure represents a perfect vacuum and any further decrease in pressure is impossible. When the velocity exceeds the value given by the above equations, the region of zero pressure expands until a cavity of appreciable size is formed within the fluid. Fully developed cavities of this kind are shown in Fig. 80. The light-colored areas indicate the vapor-filled cavities formed when water breaks away from the sides of the diverging portion of a Venturi

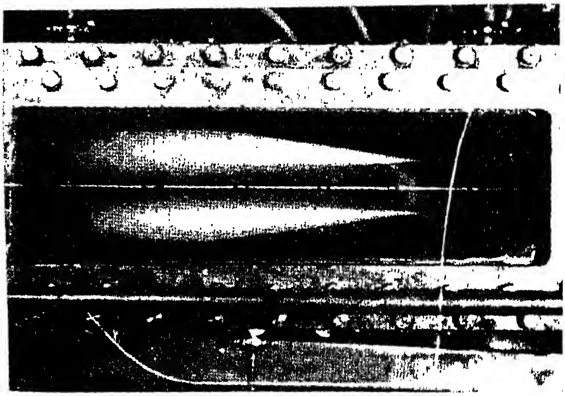


FIG. 80.—Cavitation in a Venturi tube. (*Hunsaker, J. C., Cavitation Research, Mech. Eng., vol.* 51, *no.* 4, *pp.* 211–216.)

tube. The white line shows the variation of pressure along the tube. The picture shows only one phase in the development of the cavity. The vapor in the cavity is immediately swept away and a new one begins to form at the throat.

In liquids there is always a small pressure in those regions in which cavitation has occurred because of the fact that, as the pressure drops, the liquid boils and the pressure within the cavity is the vapor pressure of the liquid surrounding it. Cavitation therefore occurs in a Venturi tube when the throat pressure p_2 is equal to the vapor pressure p_v. The corresponding velocity of flow in the straight part of the pipe would then be

$$V_{1c} = \sqrt{\frac{2g}{n^2 - 1}\left(\frac{p_1 - p_v}{w}\right)} \tag{35}$$

The subscript c has been used as an indication that this is the velocity at which cavitation begins. The corresponding velocity at the throat is

$$V_{2c} = nV_{1c} \tag{36}$$

It will be noted from Eq. (35) that the velocity at which cavitation begins in a Venturi tube is inversely proportional to the square root of the specific weight of the fluid. Then if all other conditions are identical, the velocity of cavitation is considerably greater for a gas than for water. However, it will be found that the velocity at which cavitation begins in a gas is usually so high that it is not permissible to treat the fluid as incompressible; the discussion of this phase of the subject will therefore be postponed to Chap. XIII, in which the flow of compressible fluids is studied in detail.

Problem 133. A cylinder containing water at 80°F. is fitted with a piston initially in contact with the water. If the piston is moved away from the water, what is the pressure in the cavity thus formed?

134. A Venturi meter installed in a 12-in. pipe line has a throat diameter of 5 in. The pressure at the inlet is 20 lb. per sq. in. gage. At what velocity in the main line will cavitation begin if the fluid is water at a temperature of 90°F.?

135. The relative velocity of the water at a certain point on the blades of a propeller is always three times the velocity of the submarine which it propels. If this point is 10 ft. below the water surface, what will be the velocity of the submarine when cavitation begins at the propeller? What will be the velocity when the depth is 40 ft.? Temperature is 50°F.

54. Effect of Cavitation on Fluid Flow.—The Venturi tube was discussed in the last article as a simple example of a device in which cavitation makes a marked difference in performance. In any hydraulic machine such as a pump or turbine or a marine propeller, cavitation may occur under certain conditions of operation which cause the local pressure at some point in the system to be reduced to the vapor pressure of the liquid. The details of the effects of cavitation on the behavior of such machines cannot be taken up here. It is characteristic of cavitation that it produces a sudden drop in the efficiency of the device in which it occurs.

As an illustration of how cavitation affects the performance of a hydraulic machine, the Venturi tube again forms a simple example. The Venturi tube may be regarded as an instrument

for the production of a low pressure at the throat and the subsequent reconversion of kinetic energy into pressure at the outlet. If no losses occurred, the Venturi tube would be 100 per cent efficient and the inlet and outlet pressures would be equal. The efficiency may be defined as the ratio of the pressure recovery

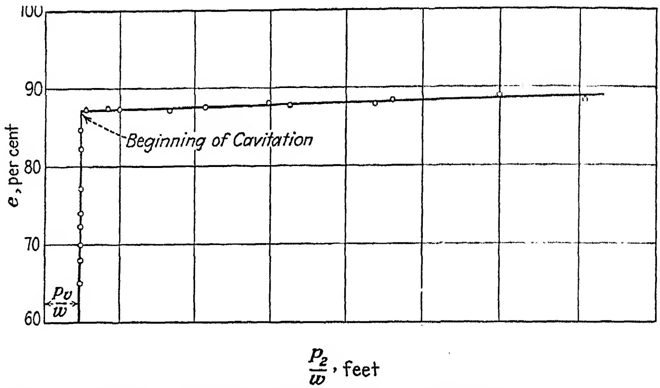


FIG. 81.—Variation of Venturi tube efficiency with throat pressure head.

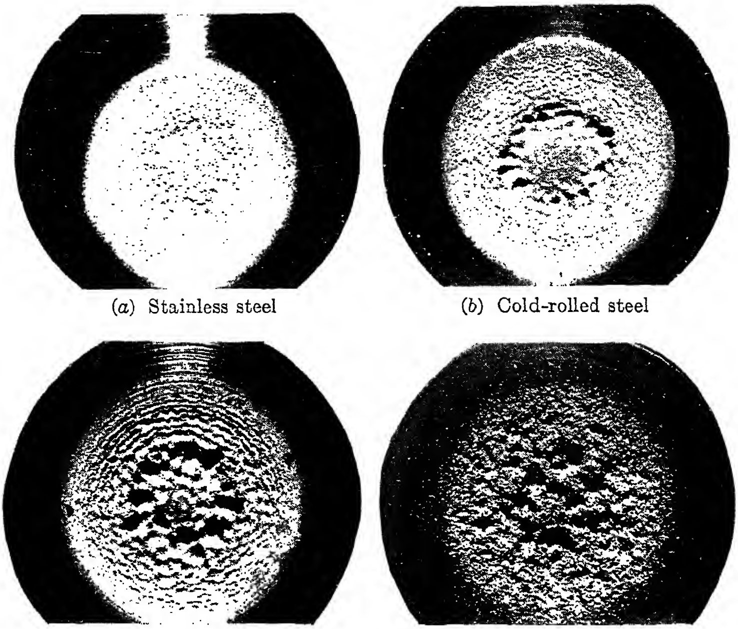
between points 2 and 3 of Fig. 79 to the pressure drop between points 1 and 2, or

$$e = \frac{\text{pressure recovery}}{\text{pressure drop}} = \frac{p_3 - p_2}{p_1 - p_2} = \frac{\frac{p_3}{w} - \frac{p_2}{w}}{\frac{p_1}{w} - \frac{p_2}{w}} \tag{37}$$

This definition has been proposed by Moody and Sorenson.[1] In their experiments on a Venturi tube, the rate of discharge was increased until the pressure at the throat corresponded to the vapor pressure p_v of the water; when this condition was reached, it was noted that the efficiency dropped almost instantaneously to much lower values than those existing at higher throat pressures. The results of a typical test run are shown in Fig. 81. Because of the marked change in the efficiency of the tube after cavitation occurs, it is apparent that there must be a considerable alteration in the nature of the flow.

[1] MOODY, LEWIS F., and ALFRED E. SORENSON, Progress in Cavitation Research at Princeton University, *Trans. A.S.M.E.*, vol. 57, no. 7, October, 1935.

55. Corrosion Produced by Cavitation.—Another extremely important condition which is found to result directly from cavitation is the corrosion or pitting of solid materials immersed in the fluid. This action is of much consequence in the operation of hydraulic machinery because it may completely destroy parts of the machine. In the past it was believed that this destruction of turbine blades and other such elements was caused primarily



(*a*) Stainless steel (*b*) Cold-rolled steel

(*c*) Brass (*d*) Cast iron

FIG. 82.—Results of oscillation tests on metal disks in water. Duration of test = 40 min.; frequency = 8500 cycles per sec.; amplitude = 0.01 mm. (*Courtesy Professor H. Peters.*)

by a chemical action resulting from the liberation of air and its constituent oxygen from solution in the water. However, it has been found that even materials which are highly resistant to oxidation are subject to the damaging effects of cavitation, and recent experimental work has shown that this action is more of a mechanical one. It appears that under certain conditions the cavities in the fluid will periodically collapse, producing an impact of fluid and solid material which causes severe deformation of the latter.

A convenient and rapid method[1] for studying the nature of the corrosion problem has been recently developed. A small disk of the material to be tested is oscillated at a high frequency in a container of fluid. In this way the nature of the cavitation corrosion produced on a given material by water or any other liquid may be found in a short time. Examples of results obtained by this method are shown in Fig. 82.

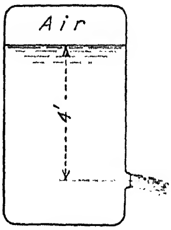


Fig. 83.

General Problems

136. A 1-in. nozzle on a $2\frac{1}{2}$-in. hose discharges a jet horizontally at a velocity of 80 ft. per sec. What is the pressure at the base of the nozzle? If, with this pressure, the velocity of discharge is 75 ft. per sec., what is the head lost in the nozzle? The fluid is water.

137. An open tank containing liquid has a small opening 4 ft. below the free surface. What is the velocity of efflux from the opening (*a*) if the liquid is water, (*b*) if the liquid is oil (specific gravity, 0.75).

138. The tank of Fig. 83 is closed and contains air at a pressure of 1 lb. per sq. in. gage above the surface of the liquid. Compute the velocity of efflux through the opening (*a*) when the liquid is all water, (*b*) when it is oil (specific gravity, 0.75), (*c*) when it is water covered with 3 ft. of oil.

139. In Fig. 84 a 12-in. pipe takes water from a reservoir at the rate of 15.7 c.f.s. What is the pressure head at *B*?

140. If the 12-in. pipe of Fig. 84 is delivering 7.85 c.f.s. and a pressure gage at *B* reads 4.6 lb. per sq. in., compute the head lost between *A* and *B*.

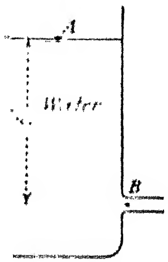


Fig. 84.

141. (*a*) Neglecting loss, compute the discharge at *F* of Fig. 85 under the conditions shown. Find the pressure at *C* and *D*. (*b*) If the discharge at *F* is 5.5 c.f.s. under the conditions shown, how much head is lost in passing into and through the tube?

142. Assuming no loss of head and neglecting vapor pressure, how much can the discharge in Fig. 85 be increased by making *EF* longer? What is then the pressure in the tube at *D*?

143. A Pitot-static tube is placed in a 6-in. pipe line carrying 3 cu. ft. of water per second. The tube is connected to a manometer which indicates a head of 3.3 in. of mercury. What is the calibration coefficient for the tube if it is to be used to indicate average velocity?

144. The cylindrical gas holder shown in Fig. 86 is closed at the top with a cover weighing 42,400 lb. which is constructed so that it descends freely

[1] Hunsaker, J. C., Progress Report on Cavitation Research at Massachusetts Institute of Technology, *Trans. A.S.M.E.*, vol. 57, no. 7, October, 1935.

when gas is withdrawn. What will be the velocity of discharge at the end of the pipe line, neglecting losses? How long will it take to draw off 8000 cu. ft. of gas? Assume gas incompressible and $w = 0.0424$ lb. per cu. ft.

145. An airship is moving through still air of standard density at a velocity of 75 m.p.h., with the axis of the hull pointing in the direction of motion. A vertical differential manometer containing alcohol (specific gravity, 0.795) is connected to a hole in the nose and one on the side of the hull. If the manometer reading is 9.65 in., what is the relative velocity of the air at the point on the side? If this point is the highest on the hull, what is the absolute velocity of the air?

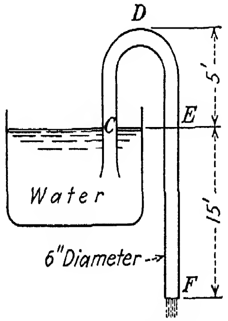


FIG. 85.—Siphon.

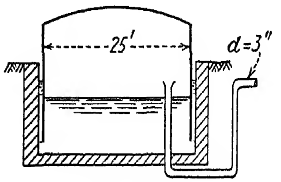


FIG. 86.

146. A Venturi meter with a ratio of diameters of 2.5 has a loss of head of 0.4 ft. between inlet and throat when the throat velocity is 20 ft. per sec. Compute the difference between inlet and throat pressures. If the loss in the diverging section is 1.2 ft., what is the total drop in pressure in the meter if the fluid is water?

147. What is the pressure at B in Fig. 87 if the tube below B is of uniform diameter? If the diameter at B is 6 in. and that at C is 4 in., compute the pressure at B and the discharge.

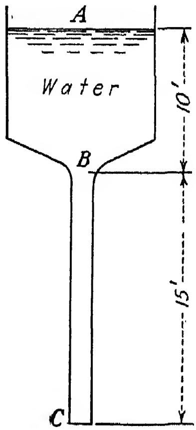


FIG. 87.

148. A horizontal water pipe 4 in. in diameter diverges gradually to 12 in. and discharges into a reservoir at a point 6 ft. below the surface. What is the discharge when the pressure is zero gage where the pipe starts to diverge?

149. A pump takes water at the rate of 2000 g.p.m. from a horizontal 12-in. pipe where the vacuum is 8 in. of mercury and delivers it at a pressure of 10 lb. per sq. in. gage to an 8-in. pipe at a point 4 ft. higher. Assuming an efficiency of 80 per cent from pipe to pipe, what power is required by the pump?

CHAPTER VI

IMPULSE AND MOMENTUM IN FLUIDS

56. Impulse and Momentum Equations. Following Newton's second law of motion, it may be stated that a mass M, having impressed upon it an effective force F, is given an acceleration a which is proportional to F and inversely proportional to M. Mass M may be expressed as weight W divided by g, the acceleration of gravity. Then the law may be written algebraically as

$$F = Ma = \frac{Wa}{g} \tag{1}$$

Expressing a in terms of increments of velocity and time, this equation becomes

$$F = M\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt} \tag{2}$$

which can be written

$$F\,dt = \frac{W}{g}dV \tag{3}$$

Integrating from time zero to time t, during which period the velocity changes from V_0 to V, and treating F as a constant, there results

$$Ft = \frac{W}{g}V - \frac{W}{g}V_0 \tag{4}$$

or

$$Ft = \frac{W}{g}(V - V_0) \tag{5}$$

The product Ft is known as the impulse of the force and the product of mass and velocity, $\frac{W}{g}V$, is the momentum of the mass or quantity of motion, the dimension of both quantities being ML/T. In words, Eq. (4) states that the change in momentum is equal to the impulse of the applied force. Letting the symbol U represent momentum, Eq. (3) may be written in the form

$$F = \frac{\left(\dfrac{W}{g}\right) dV}{dt} \qquad \frac{dU}{dt} \tag{6}$$

and Eq. (4) may be written as

$$F = \frac{\left(\dfrac{W}{g}\right) V - \left(\dfrac{W}{g}\right) V_0}{t} \qquad \frac{U - U_0}{t} \tag{7}$$

From these equations it is seen that the effective force is equal to the rate of change of momentum.

Impulse and momentum are vector quantities, just as force and velocity, and it can therefore be shown that, even with F, V and V_0 in different directions, Eq. (4) may be written as a vector equation, that is,

$$\overline{Ft} = \left(\frac{\overline{WV}}{g}\right) - \left(\frac{\overline{WV_0}}{g}\right) \tag{8}$$

the bar over a quantity being used here to indicate a vector. Then taking components along any arbitrarily chosen x- and y-axes,

$$\begin{aligned} (Ft)_x &= \left(\frac{WV}{g}\right)_x - \left(\frac{WV_0}{g}\right) \\ (Ft)_y &= \left(\frac{WV}{g}\right)_y - \left(\frac{WV_0}{g}\right)_y \end{aligned} \tag{9}$$

Equations (9) are equivalent to

$$\begin{aligned} F_x t &= \frac{W}{g}(V_x - V_{0_x}) \\ F_y t &= \frac{W}{g}(V_y - V_{0_y}) \end{aligned} \tag{10}$$

which form is usually preferred because it is convenient to deal with the change of velocity represented by the binomial in Eq. (5) and Eqs. (10) rather than with the change in momentum of Eqs. (9).

57. Conservation of Momentum.—It follows from the laws of motion that a mass cannot undergo a change of momentum except by the application of an external force. This is also true

for any system of masses. It can be demonstrated that the total momentum within a system is unchanged by an exchange of momentum between two or more masses of the system. This latter fact, known as the principle of conservation of momentum, will be demonstrated for the simple case of two masses moving in the same direction.

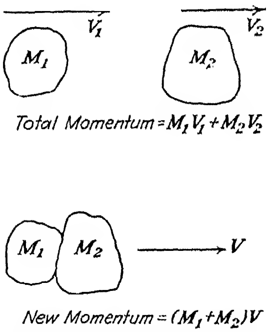


FIG. 88.—Conservation of momentum.

In Fig. 88 the mass M_1, moving with velocity V_1, overtakes the mass M_2, which has a smaller velocity V_2. While the bodies are making or breaking contact there is a force toward the left on M_1 and an equal and opposite force on M_2. However complicated the variation of these forces may be, they are at all times equal. The impulse during time Δt is $F\,\Delta t$. The total impulse, $\Sigma(F\,\Delta t)$, acting toward the left on M_1 is then equal in magnitude and opposite in direction to that on M_2. The new momentums of M_1 and M_2 are $[M_1V_1 - \Sigma(F\,\Delta t)]$ and $[M_2V_2 + \Sigma(F\,\Delta t)]$, respectively, and the total momentum after impact is then

$$[M_1V_1 - \Sigma(F\,\Delta t)] + [M_2V_2 + \Sigma(F\,\Delta t)] = M_1V_1 + M_2V_2 \quad (11)$$

Thus the total momentum after impact is the same as before impact. This is true for any number of masses moving in any direction; in general

$$\overline{M_1V_1} + \overline{M_2V_2} + \cdots + \overline{M_nV_n} = \overline{M_1V_1'} + \overline{M_2V_2'} + \cdots + \overline{M_nV_n'} \quad (12)$$

in which V and V' are, respectively, the velocities before and after impact. This is a vector equation; therefore a similar equation may be written for the components of momentum taken in any convenient direction.

If the two masses discussed above are of such materials that there is no rebound after impact, they will move along together with a velocity V, which can be computed by equating the original momentum to the new momentum. That is

$$(M_1 + M_2)V = M_1V_1 + M_2V_2 \quad (13)$$

After computing V, the loss in kinetic energy due to impact can be found, it being merely the difference between the initial and final kinetic energies of the masses. The transfer of momentum from mass to mass is always accompanied by a loss in kinetic energy. This fact largely accounts for the energy required to maintain the flow of a fluid.

58. Momentum of a Stream.—The equations of momentum are readily applied to problems dealing with flowing fluids. In such problems the mass considered may be the mass discharged in any convenient time t. The volume per second being Q, the volume in time t is Qt, the total weight involved is Qwt and the mass is Qwt/g. Equation (5), for example, then becomes

$$Ft = \frac{Qwt}{g}(V - V_0)$$

and the components of force obtained from Eqs. (10) are

$$\left.\begin{aligned} F_x &= \frac{Qw}{g}(V_x - V_{0_x}) \\ F_y &= \frac{Qw}{g}(V_y - V_{0_y}) \end{aligned}\right\} \qquad (14)$$

The forces F_x and F_y are components of the force acting on the mass having its momentum changed, namely, the fluid mass. The force acting upon whatever object produces the change is equal and opposite, the components being

$$P_x = \frac{Qw}{g}(V_{0_x} - V_x) \qquad (15)$$

$$P_y = \frac{Qw}{g}(V_{0_y} - V_y) \qquad (16)$$

The quantity Qw/g in Eqs. (15) and (16) being the mass per second, the right-hand member is the change in momentum per second. It may therefore be stated that the force exerted on an obstruction by a moving stream is the rate of change of momentum produced by the obstruction.

59. Forces Exerted on Fixed Surfaces by Open Jets.—An open jet which is deflected from its course by a fixed curved surface is shown in Fig. 89. If it is assumed that the jet is tangent to the surface at the point of first contact so that there

is no loss in shock, and also that the surface is very smooth so that there is no reduction in velocity as the jet moves along the surface, the forces can be computed from the momentum equations. With these assumptions the velocity is changed in direction only, the jet being deflected through an angle θ. Then V, the initial velocity, and V', the final velocity, are numerically

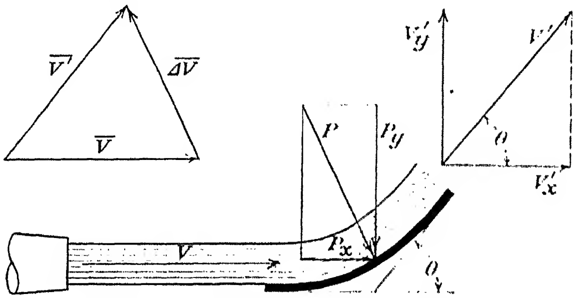


FIG. 89.—Jet deflected by a fixed surface.

the same. Taking the x-axis as parallel to the jet, the velocity components of the jet leaving the surface are

$$V_x' = V' \cos \theta = V \cos \theta$$

and

$$V_y' = V' \sin \theta = V \sin \theta$$

The components of force on the surface following Eqs. (15) and (16) are then found to be

$$P_x = \frac{Qw}{g}(V - V \cos \theta) = \frac{QwV}{g}(1 - \cos \theta) \tag{17}$$

$$P_y = \frac{Qw}{g}(-V \sin \theta) = -\frac{QwV}{g} \sin \theta \tag{18}$$

In the latter equation the negative sign indicates a downward force on the surface since the upward component of velocity was taken as positive. The amount and direction of the resultant P are found from P_x and P_y. The resultant may be found directly by first finding the vector change in velocity, which in this case is vector V – vector V', or, in the vector diagram, $\overline{V} - \overline{V'} = \overline{\Delta V}$. The resultant force P is then

$$P = \frac{Qw\,\Delta V}{g} \tag{19}$$

the direction being parallel to ΔV. This can now be divided into P_x and P_y if desired.

The flat plate at right angles to a jet shown in Fig. 90 is a special case in which θ is a right angle. The jet is divided equally by the plate so that the net change of momentum at right angles to the jet is zero. Parallel to the jet the change in velocity is V and the force normal to the plate is

$$P = QwV \qquad (20)$$

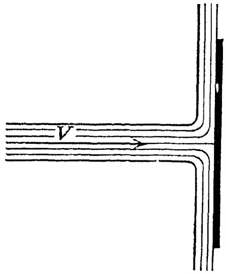
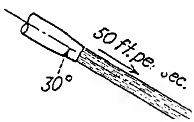


FIG. 90.—Jet deflected by a normal flat plate.

Example.—Two cubic feet of water per second are discharged from above into a tank. The jet is at an angle of 30 deg. with the horizontal and has a velocity of 50 ft. per sec. Find the components of force exerted by the jet on the tank if the latter is moving horizontally toward the right at a velocity of 15 ft. per sec.



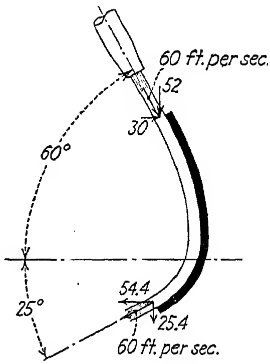
Solution.—The mass having its momentum changed each second is $2 \times 62.4/32.2$ slugs. Then the change in vertical velocity is from 25 ft. per sec. to zero and the vertical force on the tank is

$$\frac{2 \times 62.4}{32.2}(25 - 0) = 96.9 \text{ lb., downward}$$

The change in horizontal velocity is from 43.3 to 15 ft. per sec. and the horizontal force on the tank is

$$P_x \quad \frac{2 \times 62.4}{32.}(43.3 - 15) =$$

$$109.7 \text{ lb., toward the right}$$



Example.—A 2-in. jet of water with a velocity of 60 ft. per sec. is deflected by a fixed curved plate as shown in the figure. What are the horizontal and vertical components of force on the plate?

Solution.—The area of the jet is $\pi/144$ sq. ft. Then the mass per second is $\frac{\pi}{144} \times 60 \times \frac{62.4}{g} =$ 2.54 slugs. The components of velocity on contacting and leaving the surface are as shown. Then the horizontal force is

$$P_x = 2.54[30 - (-54.4)] = 214 \text{ lb., toward the right}$$

During the same time the vane moves from C to D with a velocity v, the distance being $v\,\Delta t$, and that part of the jet which is deflected by the vane is the prism DE having a volume

$$A(V - v)\Delta t.$$

Letting Q' represent the rate at which the liquid overtakes the vane, then

$$Q'\,\Delta t = A(V - v)\Delta t \tag{25}$$

and from Eqs. (24) and (25)

$$Q' = \frac{Q(V - v)}{V} \tag{26}$$

The mass having its momentum changed by the single vane in unit time is $Q'w/g$.

A series of vanes can be so arranged that the entire discharge of the jet is deflected by the vanes. The rate at which the liquid is being deflected is then Q, the total discharge of the jet. This is usually accomplished by vanes correctly spaced on the periphery of a wheel.

62. Forces Exerted by Jets on Moving Vanes. In Fig. 95 the vane has a velocity v in the same direction as the jet that is tangent to the vane at the point of contact. The jet strikes the

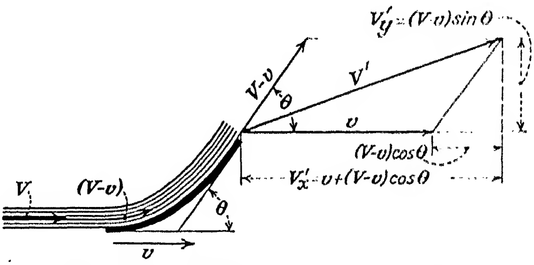


FIG. 95.—Jet deflected by a moving vane.

vane with a relative velocity of $V - v$ and, assuming no friction along the surface, the relative velocity is constant in magnitude, being changed only in direction. The absolute velocity V' of the jet leaving the vane is the vector sum of its velocity relative to the vane, $(V - v)$, and v, the velocity of the vane. The addition of this relative velocity and vane velocity to produce V' is indicated by the vector diagram of Fig. 95. From the geometry

of this figure the components of V' are $V_x' = v + (V - v)\cos\theta$ and $V_y' = (V - v)\sin\theta$. Substituting these components in Eqs. (15) and (16), in which the forces on the deflecting surface are shown to be equal and opposite to the rate of change of momentum of the fluid, there results for the components of the total force on a series of vanes

$$P_x = \frac{Qw}{g}\{V - [v + (V - v)\cos\theta]\}$$

or

$$\frac{Qw}{}(V - v)(1 - \cos\theta) \tag{27}$$

and

$$P_y = -\frac{Qw}{g}(V - v)\sin \tag{28}$$

The force on a single vane is found by replacing the Q of Eqs. (27) and (28) with $Q' = Q(V - v)/V$ from Eq. (26), with the result:

$$P_x = \frac{Qw}{g}\frac{(V - v)^2}{V}(1 - \cos\theta) \tag{29}$$

$$P_y = -\frac{Qw}{g}\frac{(V - v)^2}{V}\sin\theta \tag{30}$$

63. Power Developed by a Series of Vanes.—In the last article the force exerted in the direction of motion by a jet having velocity V on a series of vanes moving with velocity v was shown to be

$$P_x = \frac{Qw}{g}(V - v)(1 - \cos\theta) \tag{31}$$

The work done on the vanes by this force in 1 sec., that is, the power, is the product of the force and the distance traveled by the vane, or

$$E \;\cdot P_x v = \frac{Qw}{g}(Vv - v^2)(1 - \cos\theta)$$

The value of θ for maximum power is found by inspection to be 180 deg. The proper velocity of the vane to produce maximum power for a given jet velocity may be found by equating dE/dv to zero. Thus

$$\frac{dE}{dv} = \frac{Qw}{g}(V - 2v)(1 - \cos\theta) = 0$$

and

$$v = \qquad (32)$$

Then the power from a series of vanes will be a maximum when $v = V/2$ and $\theta = 180$ deg. Suppose an attempt is made

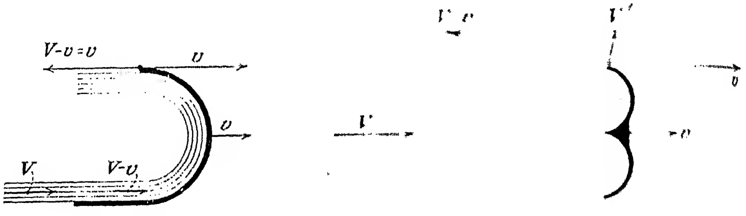


FIG. 96.—Vane velocity for maximum power.

to realize both of these conditions, as in Fig. 96a. The velocity of the jet relative to the vane is $V - v = v$. The final velocity of the water is then the velocity v of the vane toward the right plus $(V - v)$, the velocity of water relative to the vane, toward the left. This sum is zero when $v = V/2$ and all the kinetic energy will have been removed. This is obviously impracticable because no fluid can be discharged at zero velocity.

In practice the angle must be made somewhat less than 180 deg. as in Fig. 96b, but the relation $v = V/2$ corresponds closely to the usual working conditions. If the jet is divided evenly by the cusped vane of Fig. 96b, the resultant transverse force is zero.

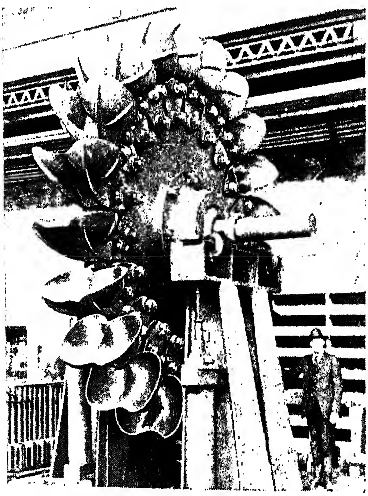
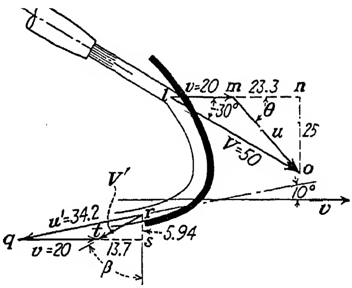


FIG. 97.—Impulse wheel for a 30,000-hp. unit. (*Courtesy Allis-Chalmers Mfg. Co.*)

The impulse wheel shown in Fig. 97 is an example of a series of cusped vanes. The speed of the wheel is such that the velocity

of the buckets under normal operating conditions is nearly $V/2$. Then if N is the r.p.m. of the wheel and d the diameter of the bucket circle, the best speed is $N = 30V/\pi d$.

Example.—A jet of water having a velocity of 50 ft. per sec. discharges 3 c.f.s. of water against a series of vanes. The vanes are moving with a velocity of 20 ft. per sec., as shown in the figure, and angle θ is such that there is no shock, that is, the water moves tangent to the vane. Compute angle θ, P_x, P_y and the horsepower and efficiency of the vanes.



Solution.—Since this is a series of vanes, the quantity striking the vanes is the total 3 c.f.s. and the mass per second is

$$3 \times 62.4 \div 32.2 = 5.81 \text{ slugs/sec.}$$

Now the velocity relation at impact is such that

Velocity of water = velocity of vane + velocity of water relative to vane

$$\overline{V} = \overline{v} + \overline{u}$$

This condition is shown graphically by the vector diagram *lmo*. By simple trigonometry $ln = 43.30$, $mn = 23.30$, $no = 25$,

$$\theta = \tan^{-1}\frac{no}{mn} = \tan^{-1}\frac{25}{23.30} = 47 \text{ deg.}$$

and $u = 34.2$ ft. per sec. If there is no loss, the relative velocity continues to be 34.2 ft. per sec. along the face of the vane, being changed only in direction, and at the discharge end there is the vector relation

$$\overline{V'} = \overline{v} + \overline{u}'$$

This condition is shown graphically by vector diagram *qrt* and from this $qs = 34.2 \cos 10° = 33.7$, $rs = 5.94$ and $V' = \sqrt{13.7^2 + 5.94^2} = 14.93$. The direction of V' is defined by $\beta = \tan^{-1}\dfrac{13.7}{5.94} = 66°30'$.

The x- and y-components of the initial velocity V and the final velocity V' can now be used to compute the forces. Then

$$P_x = 5.81[43.3 - (-13.7)] = 331 \text{ lb.}$$
$$P_y = 5.81[-25 - (-5.94)] = -111 \text{ lb.}$$

The work done by P_x in 1 sec. is force times distance, or $331 \times 20 = 6620$ ft. lb. per sec. This can be checked by taking the difference between the kinetic energy of the water as it strikes and as it leaves the vanes. Thus

$$\text{Initial K.E. per sec.} = \frac{1}{2}\frac{Qw}{g}V^2 = \frac{5.81}{2} \times 50^2 = 7260 \text{ ft.lb.}$$

$$\text{Final K.E. per sec.} = \frac{1}{2}\frac{Qw}{g}V'^2 = \frac{5.81}{2} \times 14.93^2 = 647 \text{ ft.lb.}$$

$$\text{Work per sec.} = \text{initial K.E.} - \text{final K.E.} = 6613 \text{ ft.lb.}$$

$$\text{Efficiency} = \frac{\text{output}}{\text{input}} = \frac{6613}{7260} = 91.1 \text{ per cent}$$

Problem 159. Suppose a single flat plate to be carried at a velocity of 10 ft. per sec. toward a nozzle that is discharging 60 g.p.m. of water with a velocity of 30 ft. per sec. At what rate in gallons per minute does the water strike the plate? What force is needed to carry the plate if it is normal to the jet?

160. A vane curved through an arc of 60 deg. receives a tangential jet of water which has a velocity of 100 ft. per sec. The vane is moving in the same direction as the jet with a velocity of 40 ft. per sec. Find the absolute velocity of the water leaving the vane, the change in kinetic energy per pound of water and the angle through which the jet is deflected.

161. A series of vanes which are curved through an arc of 135 deg. is moving with a velocity of 50 ft. per sec. They receive 2 c.f.s. in a tangential jet which has a velocity of 120 ft. per sec. What are the forces parallel and normal to the axis of the jet? What power is developed.

162. The cusped vanes on an impulse wheel are curved through an arc of 160 deg. The wheel is driven at its best speed, 240 r.p.m., by a 2-in. jet of water having a velocity of 300 ft. per sec. Compute the force on the vanes and the power developed.

163. For a single vane, what is the relation of vane velocity to jet velocity for maximum power? Use the method employed in deriving Eq. (32).

64. Pressure on Pipe Bends.—The interior surface of a bend in a pipe line is subjected to a pressure due in part to the normal pressure under which the line operates and in part to the fact that the direction of the momentum is changed. The body of fluid within a bend which turns through an angle of θ is shown in Fig. 98*a*. It is subjected to pressure forces on all sides, the force R being the resultant pressure force on the fluid at the walls of the bend and the force pA on each end being the pressure force exerted by the adjacent fluid. For convenience, components of forces and velocities are taken parallel and normal to OB, the bisector of the angle θ. Then letting the applied force equal the change of momentum per second, and noting that there is no net change of momentum normal to OB,

$$R - 2pA \sin\frac{\theta}{2} = \frac{2Qw}{g}V \sin\frac{\theta}{2}$$

or

$$R = 2pA \sin \frac{\theta}{2} + \frac{2Qw}{g} V \sin \frac{\theta}{2} \tag{33}$$

If the bend is not supported in its plane in any other way, the force on it must be balanced by a tensile force T in the pipe line as shown in Fig. 98*b*, in which R is the pressure force on the bend. The force T can be determined by again considering components

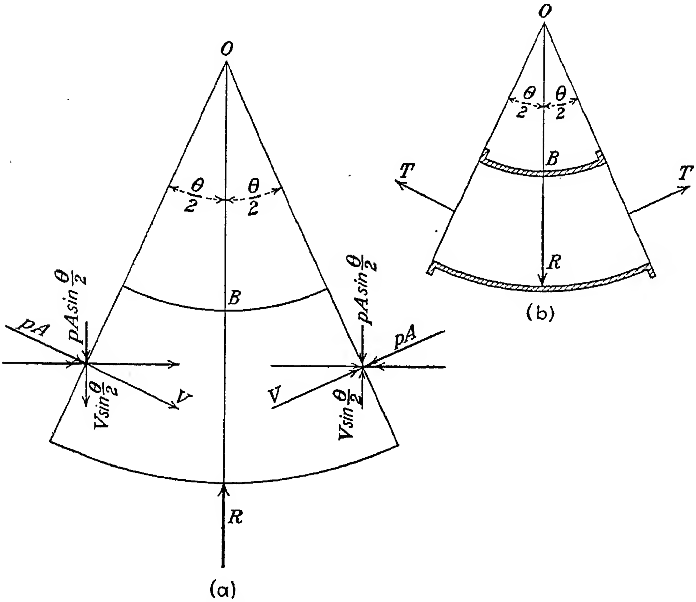


FIG. 98.—Forces on a pipe bend.

parallel to OB, using the value of R from Eq. (33) and writing $\Sigma F_{OB} = 0$, whence

$$2T \sin \frac{\theta}{2} = R = 2 \sin \frac{\theta}{2}\left(pA + \frac{QwV}{g}\right)$$

and

$$T = pA + \frac{QwV}{g} \tag{34}$$

It appears from this equation that T is independent of the angle and that any curvature, however small, of an unsupported line results in a tensile force.

65. Forces on Reducing Bends and Reducers. The pressure forces on the interior of a reducing bend can be found by considering the forces on the body of fluid within the bend. This body, shown in Fig. 99*a*, has pressure forces on the ends exerted by the adjacent fluid and a pressure force exerted by the side walls, the resultant of which is R. Suppose the liquid to enter the bend from the left with velocity V_1 and to leave with a velocity V_2 at an inclination of θ. Then the original momentum is QwV_1/g and the components of the final momentum are $(QwV_2 \cos \theta)/g$ and $(QwV_2 \sin \theta)/g$.

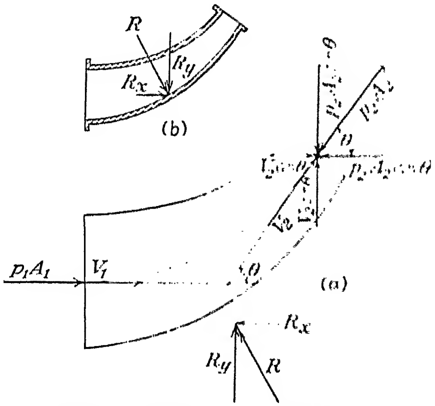


FIG. 99.—Forces on a reducing bend.

Equating the effective forces on the fluid to the rate of change of momentum and writing separate equations involving the x- and y-components gives

$$p_1A_1 - p_2A_2 \cos \theta - R_x = \frac{Qw}{g}(V_2 \cos \theta - V_1) \qquad (35)$$

and

$$-p_2A_2 \sin \theta + R_y = \frac{Qw}{g}V_2 \sin \theta \qquad (36)$$

In these equations R_x and R_y are forces acting on the fluid and the forces acting on the bend are equal and opposite. Then solving for R_x and R_y, the forces on the bend in Fig. 99*b* are found to be

$$R_x = \frac{Qw}{g}(V_1 - V_2 \cos \theta) + (p_1A_1 - p_2A_2 \cos \theta) \qquad (37)$$

$$R_y = \left(\frac{Qw}{g}V_2 + p_2A_2\right) \sin \theta \tag{38}$$

In Eq. (37) a positive R_x is a force toward the right on the bend and in Eq. (38) a positive R_y is a downward force on the bend.

The equation for the case of a simple reducer with no bend, such as shown in Fig. 100, is readily obtained by letting the angle θ be zero in Eq. (37). Then

$$R_x = \frac{Qw}{g}(V_1 - V_2) + (p_1A_1 - p_2A_2) \tag{39}$$

The force on the reducer will always act toward the smaller end. Thus the force on a nozzle placed on the end of straight line of pipe or hose is always such as to pull on the line and place it in tension. This can be readily seen without the aid of Eq. (39) by inspection of the internal and external pressures on the nozzle.

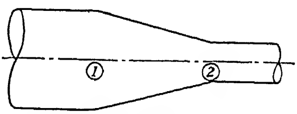


FIG. 100.—Pipe reducer.

Problem 164. What is the tensile force in a 6-in. pipe line caused by an unsupported 90-deg. bend if it is under a pressure of 40 lb. per sq. in. (a) if there is no flow, (b) if the flow of water is 10 c.f.s.?

165. The end of a horizontal 12-in. pipe is fitted with a tube which diverges gradually to 18 in. in diameter. It discharges 20 c.f.s. of water into the atmosphere. Determine by inspection of the pressures the direction of the force on the end of the 12-in. pipe. Compute the amount of this force.

166. A 60-deg. bend reduces a pipe line from 12 to 6 in. The pressure in the 12-in. line is 25 lb. per sq. in. when the flow is 10 c.f.s. Find the force in the bend parallel and normal to the 12-in. line. Does this force change if the direction of flow is reversed, the pressure in the 12-in. line being unchanged and losses being neglected?

66. Work Done in a Rotating Channel.—Figure 101 represents a rotating channel such as one of the water passages in a reaction turbine. It is rotating about an axis through O with an angular velocity ω, so that the velocities of point 1, the entrance end, and point 2, the exit, are $u_1 = \omega r_1$ and $u_2 = \omega r_2$, respectively. Water is caused to enter at the outer end with an absolute velocity of V_1 and a velocity relative to the channel of v_1. By maintaining the proper relation between V_1, u_1 and the angle α_1, the velocity v_1 is in the direction of the tube. A similar situation prevails at point 2 where the absolute and relative velocities are V_2 and v_2,

respectively. At points 1 and 2 or at any point in the channel the absolute velocity of the water is the vector sum of u, the velocity of the vane, and v, the velocity relative to the vane; that is, $\overline{V} = \overline{u} + \overline{v}$ as indicated by the vector diagrams at points 1 and 2.

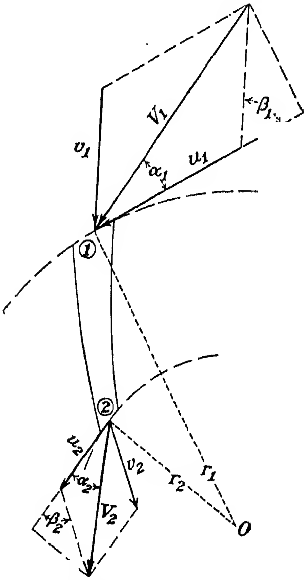


FIG. 101.—Velocity diagram for a rotating channel.

The tangential components of V_1 and V_2 are $V_1 \cos \alpha_1$ and $V_2 \cos \alpha_2$, and the corresponding tangential components of momentum of the discharge Q are $\frac{Qw}{g} V_1 \cos \alpha_1$ and $\frac{Qw}{g} V_2 \cos \alpha_2$.

It is known that the torque T applied to a rotating mass is equal to the rate of change of angular momentum, or rate of change of moment of momentum, of the mass. The moment of momentum is the product of the tangential component of momentum and the radius. The radial component has no moment. Then in this case the moments at points 1 and 2 are $\frac{Qw}{g}(V_1 \cos \alpha_1)r_1$ and $\frac{Qw}{g}(V_2 \cos \alpha_2)r_2$. The applied torque on the channel is then

$$T = \frac{Qw}{g}(V_1 \cos \alpha_1)r_1 - \frac{Qw}{g}(V_2 \cos \alpha_2)r_2 \tag{40}$$

The work done by a torque is the product of the torque and angle through which it works. In this case the work done on the channel by each pound of water is

$$\text{Work per lb.} = \frac{T\omega}{Qw} = \frac{1}{g}[(V_1 \cos \alpha_1)r_1\omega - (V_2 \cos \alpha_2)r_2\omega] \tag{41}$$

This represents the input of energy to the channel per pound of water, or feet of head taken from the water by the channel. It may be written as

$$h_I = \frac{1}{g}[(V_1 \cos \alpha_1)u_1 - (V_2 \cos \alpha_2)u_2] \tag{42}$$

Bernoulli's equation for points 1 and 2, corrected for the energy lost and for the work done by the water on the channel, is

$$\frac{V_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{w} + z_2 + h_I + h_l \tag{43}$$

From the vector diagrams

$$V_1^2 = u_1^2 + v_1^2 + 2u_1v_1 \cos \beta_1$$
$$V_2^2 = u_2^2 + v_2^2 + 2u_2v_2 \cos \beta_2$$

and

$$V_1 \cos \alpha_1 = u_1 + v_1 \cos \beta_1$$
$$V_2 \cos \alpha_2 = u_2 + v_2 \cos \beta_2$$

Equation (43), after substituting the value of h_I from Eq. (42), inserting the above values of V^2 and $V \cos \alpha$ and simplifying, becomes

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2 + \frac{u_1^2 - u_2^2}{2g} + h_l \tag{44}$$

In this form it is known as Bernoulli's equation for rotating channels. The expression $(u_1^2 - u_2^2)/2g$ is commonly called the centrifugal head.

General Problems

167. A series of vanes of the form shown in Fig. 96*b* have cross sections which curve through an arc of 135 deg. on one side of the cusp and 170 deg. on the other side. What is the axial thrust on the wheel having these vanes if it is driven at best speed by a jet of water of 2 c.f.s. having a velocity of 300 ft. per sec.?

168. The tank of Fig. 102 is discharging 3 c.f.s. of water at a velocity of 16 ft. per sec. Neglecting friction, what force is required to keep the tank at rest?

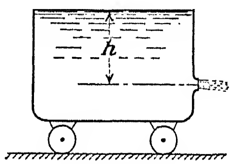


FIG. 102.

169. If the tank of Fig. 102 is discharging 3 c.f.s. of water at a velocity relative to the tank of 16 ft. per sec. and if it is moving toward the left with a uniform velocity of 6 ft. per sec., what is the force on the car? What is the head h, neglecting losses?

170. A nozzle discharges 3 c.f.s. at a velocity of 100 ft. per sec. and the jet impinges tangentially on vanes which curve through an arc of 120 deg. The vanes have a velocity of 40 ft. per sec. parallel to the jet. Compute the

absolute velocity of the water leaving the vanes, the forces parallel and normal to the jet, the change in kinetic energy of 1 lb. of water and the power developed.

171. A jet of water of 200 g.p.m. with a velocity of 40 ft. per sec. strikes a series of vanes as shown in Fig. 103 and leaves them with an absolute velocity of 20 ft. per sec. in the direction shown. What forces parallel and normal to the jet are acting on the vanes? What power is developed?

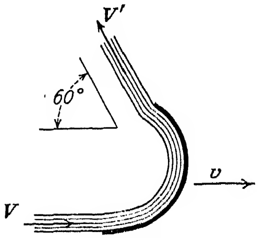


FIG. 103.

172. A boat is propelled at a speed of 15 m.p.h. by a pump which takes in water and discharges it at the rate of 10 c.f.s. through a 6-in. pipe at the stern. What propelling force is exerted on the boat? Compute the work per second done on the boat, the kinetic energy in the water discharged each second and the efficiency of the propelling system, assuming the pump to be 100 per cent efficient.

173. A body shown in Fig. 104 is moved toward the jet of water with a velocity of 20 ft. per sec. The jet has a discharge of 1 c.f.s. at a velocity of 60 ft. per sec. Find the power required to move the body and the absolute velocity of the water as it leaves.

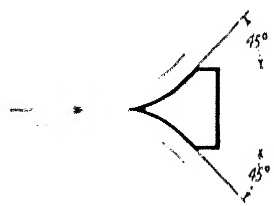


FIG. 104.

174. A jet of water of 3 c.f.s. at a velocity of 100 ft. per sec. strikes a series of vanes as in Fig. 105. The relative velocity of the jet is tangent to the moving vanes. Find the angle α, the work done per second and the efficiency. What is the force normal to v, the velocity of the vane?

175. A horizontal 12-in. pipe bends 90° into a horizontal 6-in. line. If the pressure in the 12-in. line is maintained at 20 lb. per sq. in., what force is required to support the bend against the action of the water (*a*) when there is no flow, (*b*) when flow is at the rate of 10 c.f.s.?

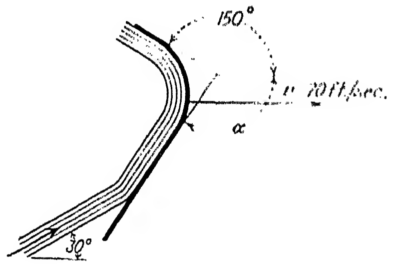


FIG. 105

176. In a machine having passages similar to that shown in Fig. 101, $V_1 = 30$ ft. per sec., $r_1 = 1.5$ ft., $r_2 = 0.75$ ft., $b_1 = 1.5b_2$, $\alpha_1 = 30$ deg., $v_1 = u_1$, $\beta_2 = 90$ deg., $p_1 = 15$ lb. per sq. in. and $Q = 50$ c.f.s. Find β_1, v_1, u_1, ω, u_2, V_2, p_2, torque and power developed. Assume head lost to be $0.2v_1^2/2g$.

CHAPTER VII

DYNAMIC LIFT AND PROPULSION

67. The Theory of Lift.—When the resistance or force produced by the motion of a body through a mass of fluid is discussed, it seems natural to think of this force as acting in such a way as to oppose the motion of the body. There are, however, a number of important cases where not only is such a resistance produced but where, in addition, the complete reaction of the fluid on the body has a component in the direction normal to that of the motion. Usually the component opposing the motion is referred to as the drag, while the cross-stream component is called the lift, even though it may not always be acting vertically upward. As examples of bodies which produce both of these force components, there are the rotating cylinder or sphere, the vanes of a turbine wheel, the sails of a ship and the wings of an airplane.

If the flow around a body is known in complete detail, so that the velocity at any point on the surface can be determined, the lift and drag components of the resultant force can be calculated by using Bernoulli's theorem to determine the corresponding pressures and then summing up the proper components of the elementary forces which they produce over the entire surface. Another method which is particularly suitable in some cases is to consider the time rate of change of momentum of the flow in the direction of the desired force component. Both of these methods will be employed in the discussions that follow.

Problem 177. A body placed in a uniform stream of air develops a resultant force of 75 lb. acting upward and toward the left at an angle of 85 deg. with the horizontal. What are the lift and drag components (*a*) when the stream of air is flowing horizontally and toward the left, (*b*) when it is flowing toward the left and upward at an angle of 10 deg. with the horizontal?

68. The Magnus Effect on Rotating Cylinders.—It was first demonstrated experimentally by Magnus, in 1852, that, if a

circular cylinder is rotated about its axis and at the same time is caused to move forward through a mass of fluid, a cross-wind force or lift is produced. The same result is obtained if the rotating cylinder is placed in a stream of fluid which moves with a uniform velocity at a great distance from the cylinder, the relative motion with respect to the body being the same in both cases. An estimate of the magnitude of this lifting force can be made if it is assumed that the cylinder is infinitely long in the direction of its axis so that the flow patterns in any two planes perpendicular to that axis are identical. Such a flow is said to be two-dimensional since all the velocity vectors representing the flow in any one of these perpendicular planes lie entirely in that plane and there are no lateral components.

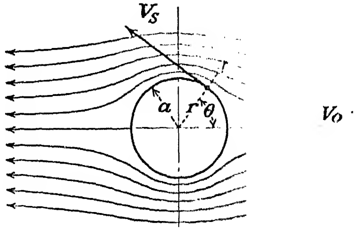


FIG. 106.—Streamlines for two-dimensional flow past a circular cylinder.

It is well known in the field of classical hydrodynamics that the flow of a uniform stream of fluid past a nonrotating circular cylinder gives rise to a system of streamlines, as shown in Fig. 106. The velocity at any point on the surface of the cylinder is found in classical hydrodynamics to be

$$V_s = 2V_0 \sin \theta \tag{1}$$

where V_0 is the velocity of the stream far ahead of the cylinder and θ is the polar angle that determines the position of the point on the circumference for which the local velocity is to be computed. The flow illustrated in Fig. 106 is represented by Eq. (1), and is set up so that its general direction is from right to left. The complete development of this flow pattern and the determination of the value of the velocity at any point are problems which are outside the scope of this work.

Now, if the pressure in the undisturbed stream is represented by p_0 and that at a point on the cylinder by p, an application of Bernoulli's theorem to these two points shows at once that

$$p = p_0 + \frac{\rho}{2}(V_0^2 - V_s^2) = p_0 + \frac{\rho V_0^2}{2}(1 - 4 \sin^2 \theta) \qquad (2)$$

Differences in elevation between the two points considered have been neglected. The pressure difference $p - p_0$ may be expres-

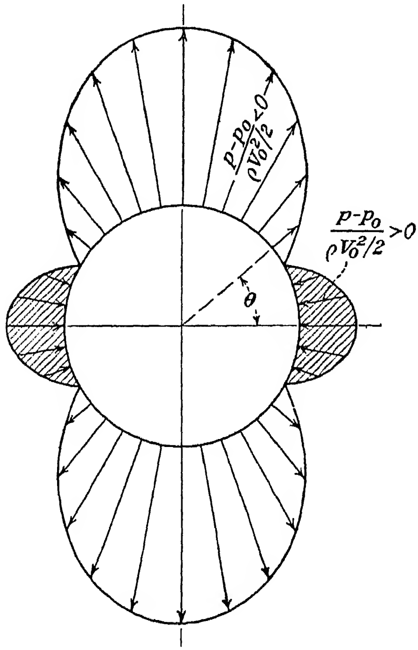


FIG. 107.—Theoretical pressure distribution on a circular cylinder.

sed as a ratio to $\rho V_0^2/2$, the dynamic pressure of the undisturbed stream, for from Eq. (2) this ratio is readily found to be

$$\frac{p - p_0}{\rho V_0^2} = 1 - 4 \sin^2 \theta \qquad (3)$$

The nature of the pressure distribution on the surface of the cylinder is shown in Fig. 107, in which the values given by Eq. (3) have been plotted along the radii corresponding to the different values of θ. The boundary of the cylinder has been taken as the line of zero pressure difference and all the values are plotted radially outward, whether positive or negative. The regions

of positive pressure are shaded, while the negative values occur in the unshaded areas. It is evident at once that, because of the complete symmetry of this diagram, there can be no resultant force in either the direction parallel to or perpendicular to that of the undisturbed stream; that is, both the lift and drag are equal to zero.

If the cylinder shown in Fig. 106 is supposed to be located in a mass of fluid which is completely at rest and if an agency is provided whereby the cylinder may be rotated about its axis, a circulatory flow such as that shown in Fig. 108 will be produced. In order to establish such a flow in a real fluid, it is necessary to consider the action of viscosity between the cylinder surface and the layers of fluid adjacent to it, but, once the flow is fully developed, it may be considered as though it were taking place in a nonviscous fluid. It is to be expected that the velocities produced in the fluid by the rotation of the cylinder will vary inversely as the distance from the center; that is,

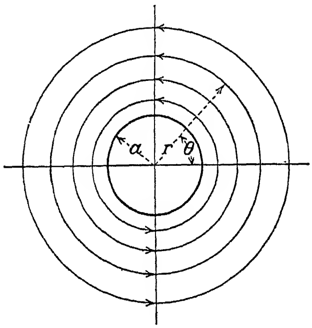


FIG. 108.—Circulatory flow around a rotating circular cylinder.

$$V = \frac{\Gamma}{2\pi r} \tag{4}$$

where Γ is a constant, r is the radius drawn to the point in question and V is the velocity perpendicular to that line. Thus, if the radius of the cylinder is represented by a, the peripheral velocity on its circumference is

$$V_p = \frac{\Gamma}{2\pi a} \tag{5}$$

The case of a cylinder rotating in a stream of moving fluid may now be considered as a combination of the two flows previously discussed. The resulting flow pattern is of the character shown in Fig. 109a, while the velocity on the circumference is now the sum of the values given by Eqs. (1) and (5), so that

$$V = 2V_0 \sin \theta + \frac{\Gamma}{2\pi a} \tag{6}$$

There is obviously an increase in velocity above the upper surface and a decrease below the lower. The pressures in these regions will therefore be lower and higher, respectively, than their values for the case of no rotation. Hence it is to be expected that there will be a force acting on the cylinder in the direction normal to that of the undisturbed stream. If this force or lift is represented by L for a unit length of the cylinder in the direction of its axis,

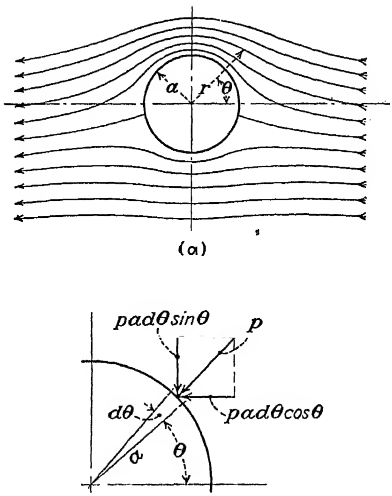


FIG. 109.—Rotating cylinder in a uniform stream.

then, as shown in Fig. 109*b*, the lift dL on an element of area $a\,d\theta$ is $-pa\,d\theta \sin\theta$. The total lift is

$$L = -\int_{-\pi}^{\pi} pa\,d\theta \sin\theta \tag{7}$$

The value of the pressure p may be obtained by using Bernoulli's theorem as in the previous example. With the introduction of the value of the local velocity obtained from Eq. (6), the result is

$$p = p_0 + \frac{\rho V_0^2}{2} - \frac{\rho}{2}\left(2V_0 \sin\theta + \frac{\Gamma}{2\pi a}\right)^2$$

When this expression is substituted in Eq. (7) and the integration is carried out, the value of the lift force is found to be

$$L = \rho \Gamma V_0 \tag{8}$$

A similar evaluation of the drag force shows it to be zero, a result which might be expected from the symmetry of the flow about the vertical axis.

Example.—A cylinder 4.2 ft. in diameter rotates about its axis in an air-stream having a velocity of 80 m.p.h. It develops a lift of 45 lb. per ft. of length. Determine the rotational speed and the location of the stagnation points, assuming that the flow conforms with the perfect-fluid theory.

Solution.—The lift per unit length is

$$L = \rho \Gamma V_0$$

from which

$$\Gamma = \frac{L}{\rho V_0}$$

Assuming the air to be of standard density,

$$\Gamma = \frac{45}{0.002378 \times (80 \times {}^{88}\!/\!_{60})} = 161.3 \text{ ft.}^2/\text{sec.}$$

From Eq. (5), $\Gamma = 2\pi a V_p$, where V_p is the peripheral speed of the cylinder. Also $V_p = 2\pi a N/60$, N being r.p.m. Hence, in terms of the diameter d,

$$\Gamma = \frac{\pi^2 d^2 N}{60}$$

from which

$$N = \frac{60\Gamma}{\pi^2 d^2} = \frac{60 \times 161.3}{(3.14 \times 4.2)^2} = 55.7 \text{ r.p.m.}$$

When $V = 0$, the value of $\sin\theta$ from Eq. (6) is

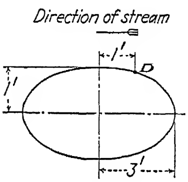


FIG. 110.

$$\sin\theta = -\frac{\Gamma}{4\pi a V_0}$$

$$= -\frac{161.3}{4 \times 3.14 \times 2.1 \times 117.2} = -0.0521$$

$$\theta = \sin^{-1}(-0.0521) = -3° \text{ or } 183°$$

Problem 178. At what points on the surface of the circular cylinder shown in Fig. 106 is the pressure difference $p - p_0$ equal to zero? What are the maximum and minimum values of the pressure difference and at what points are they found?

179. A circular cylinder 2 ft. in diameter is rotated about its axis in a mass of fluid initially at rest. If the velocity at a point 3 ft. from the surface is 15 ft. per sec., what is the peripheral speed of the surface of the cylinder?

180. What values of θ (see Fig. 109) determine the location of the stagnation points of the flow of a uniform stream past a rotating circular cylinder? What is the value of the speed ratio V_p/V_0 when these points coincide?

181. A cylinder of elliptical cross section is placed in a uniform stream of fluid, as shown in Fig. 110. The difference in pressure between point B and

the undisturbed stream is -0.47 lb. per sq. in. Determine the magnitude and direction of the components of this pressure acting parallel to the lift and drag axes of the cylinder.

69. The Circulation.—It will be noted that the constant Γ appears in the expression for the lift force acting on the rotating cylinder. It is now necessary to investigate the physical significance of this quantity. The value of Γ may be determined from Eq. (5) for the velocity due to the circulatory flow, as

$$\Gamma = 2\pi a V_p \tag{9}$$

This quantity is therefore equal to the product of the peripheral velocity of the cylinder and its circumference, and is known as the circulation. In its most general form, the circulation around a closed curve is defined as the integral of the product of the tangential component of velocity multiplied by the corresponding elements of length of the path. Hence

$$\Gamma = \int V \cos (V,ds)ds \tag{10}$$

in which (V,ds) denotes the angle between the velocity and the tangent to the curve. This definition is analogous to that of work, for, if the velocity vector is replaced by a force and the element ds is considered as a displacement of a mass particle, then Eq. (10) would be equivalent to the general definition of work in mechanics. The value of the lift per unit length of the circular cylinder, as given by Eq. (8), may now be said to be equal to the product of the density of the fluid, the circulation around the cylinder and the velocity of the stream far ahead of it. This relationship is known as the Kutta-Joukowski theorem and will later be shown to be true for a cylinder of any cross section, provided there is a circulation Γ around it.

70. The Lift Coefficient.—If the value of the circulation around the circular cylinder is determined on the basis of Eq. (9), then from Eq. (8) the expression for the lift force per unit of length may be written as

$$L = 2\pi\rho a V_p V_0 \tag{11}$$

Following aeronautical practice, it is convenient to express this force in terms of the lift coefficient C_L, that is,

$$L = C_L \frac{\rho V_0^2}{2} S \tag{12}$$

where S represents a characteristic area of the body. For bodies which produce a lift force, this area is usually taken as that of the projection of the body on a plane normal to the direction of the lift vector. It should be noted here that the lift force is always taken as perpendicular to the direction of the velocity of the undisturbed stream far ahead of the body. From Eq. (12) it is apparent that the lift coefficient has the value

$$C_L = \frac{L}{\frac{\rho V_0^2}{2} S} \tag{13}$$

so that it may be regarded as the ratio between the actual lift on the body and the force that would be produced if the dynamic pressure of the stream acted at every point of its projected area.

In the case of the rotating cylinder, the area S for a span of unit length is numerically equal to the diameter $2a$, so that the lift coefficient is

$$C_L = \frac{2\pi\rho a V_p V_0}{\frac{\rho V_0^2}{2} \cdot 2a} = 2\pi \frac{V_p}{V_0} \tag{14}$$

Thus, the lift coefficient is directly proportional to the ratio of the peripheral and forward speeds of the cylinder, this theoretical result being shown in graphical form in Fig. 111.

It is now of interest to compare the theoretical value for C_L given by Eq. (14) with the results obtained experimentally. In order to duplicate experimentally the conditions that have been set up in the theoretical development, it is necessary to find some way of obtaining a two-dimensional flow, since it is, of course, impractical to make the cylinder of infinite length. This has been accomplished with good success by attaching plates in the form of circular disks to the ends of the cylinder so as to eliminate as far as possible any lateral flow that would otherwise take place in these regions.

Tests have been made in this way in an airstream by Ackeret,[1] and his results are shown by the experimental curve in Fig. 111. The agreement is not at all satisfactory, the principal reason

[1] MÜLLER, W., "Mathematische Strömungslehre," p. 118. Julius Springer, Berlin, 1928.

for the discrepancy being that the theoretical value of the circulation is not developed in the actual flow. For the values of V_p/V_0 between zero and unity, it appears that there is an appreciable lag in the transfer of the circulatory motion from the cylinder to the surrounding fluid. In the range from $V_p/V_0 = 1$ to about 3, the lift coefficient increases in approximately a linear manner but the rate is considerably less than the theoretical value of 2π. Beyond $V_p/V_0 = 3$, the rate of increase of C_L becomes considerably less and, while the experimental curve shown does not give a definite maximum value, it appears that a value of C_L between 9 and 10 is the highest obtainable and would

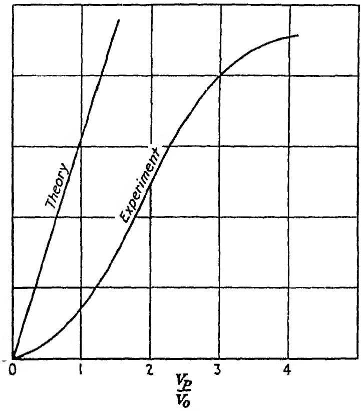


FIG. 111.—Variation of lift coefficient with speed ratio for a rotating cylinder.

correspond to a speed ratio between 4 and 5. In general, it can be said that all these differences are due directly or otherwise to the effects of the viscosity. In spite of the fact that there are large differences between theory and experiment, it should not be considered that the theory is entirely worthless, because it does give a satisfactory explanation of the general nature of the flow and presents a good qualitative representation of it.

Problem 182. A cylinder 2 ft. in diameter is rotating about its axis at 120 r.p.m. in an airstream having a velocity far ahead of the cylinder of 25 m.p.h. What are the values of the lift per foot of length of the cylinder and the theoretical lift coefficient?

183. Determine the values of $\dfrac{(p - p_0)}{\rho V_0^2/2}$ for points on the surface of a circular cylinder rotating in a uniform stream when the speed ratio V_p/V_0

is 1.5. Take values of θ in increments of 10 deg. and plot the results in the manner used in Fig. 107.

184. What is the lift coefficient for a rotating cylinder 6 in. in diameter if the lift force on it is 400 lb. per ft. of length when it is moving through water with a velocity of 20 ft. per sec.? What is its rotational speed in r.p.m., assuming the Kutta-Joukowski theorem holds?

185. A cylinder 1.5 ft. in diameter is rotated at 250 r.p.m. in an airstream having a velocity of 100 m.p.h. and develops an actual lift of 18 lb. per ft. of length. What is the ratio between the actual and theoretical lifts?

71. The Lifting Vane.—There are bodies, such as airplane wings, which are capable of producing a lift force when moved through a mass of fluid. Such bodies may be considered under the general classification of lifting vanes and it is now proposed

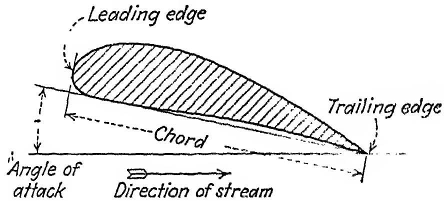


FIG. 112.—Cross section of a typical lifting vane.

to discuss the theory of lift as applied to bodies of this kind. As in the case of the rotating cylinder, the lifting vane will be considered as a cylinder of infinite length placed in a stream of fluid perpendicular to the generators of the cylinder. A typical cross section of such a vane is shown in Fig. 112. In determining the shape of the section, some arbitrary line, usually drawn tangent to the undersurface and through the rear edge, is chosen as a base line. The projection of the vane section on the base line is known as the chord, and the base line is frequently referred to as the chord line. The inclination of the section is used in evaluating the forces acting on it and is determined by the value of the angle between the chord line and the direction of the undisturbed stream of fluid. This angle is called the angle of attack. The forward or nose portion of the section is called the leading edge, while the rear portion, which is frequently sharp, is known as the trailing edge.

In developing an expression for the lift force acting on this cylinder, it is convenient to consider first a row consisting of an infinite number of such blades or vanes of identical shape, as shown in Fig. 113. These vanes are placed at the same angle

of attack, with the forward edges on a line MN perpendicular to the direction of the stream of fluid and uniformly spaced along the row at a distance h apart. With this arrangement the flow pattern will repeat itself periodically, the distance between corresponding streamlines being equal to the blade spacing. The lines AB and CD are to be regarded as two such streamlines, the lines AC and BD being drawn parallel to MN. The average velocities over the lines AC and BD are represented by V_1 and

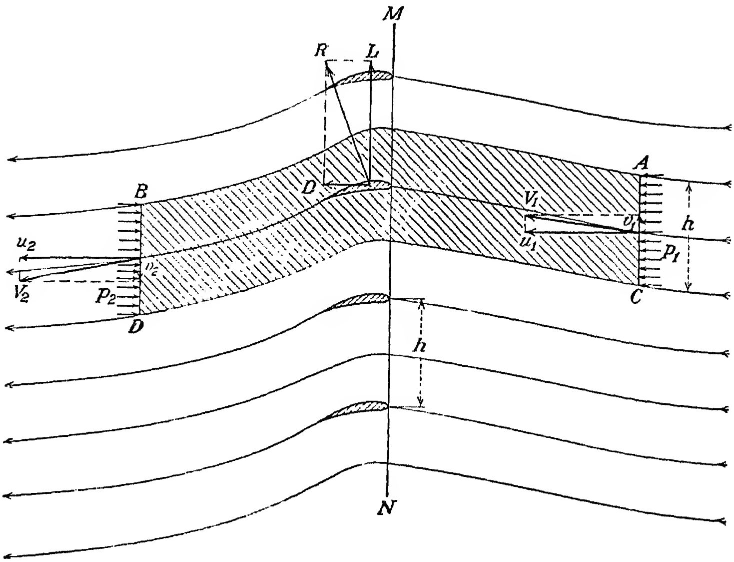


FIG. 113.—A row of lifting vanes.

V_2, respectively, their components in the direction perpendicular to MN being u_1 and u_2, while those parallel to MN are v_1 and v_2. The corresponding average pressures are p_1 and p_2.

It is now proposed to study the relationships between the various forces acting on the fluid enclosed within the area $ABDC$, a slice of unit thickness in the direction normal to the cylinder generators being selected for this purpose. In considering those components acting parallel to MN, upward forces and velocities will be treated as positive. There are no forces acting in this direction on the ends AC and BD, and any forces due to pressures acting on the streamlines AB and CD will exactly balance because of their identical shape. There is no flow across the

streamlines AB and CD, but there is a flow through ends AC and BD which involves a change in momentum. The mass of fluid entering the space $ABDC$ in unit time through the face AC is

$$m = \rho h u_1$$

and the vertical component of its momentum, in the upward direction, is

$$M_1 = \rho h u_1 v_1$$

Considering the flow to be steady, the same mass must leave through the face BD and its vertical momentum is

$$M_2 = \rho h u_1 v_2$$

The density is considered constant throughout. The force required to produce the change in momentum of the fluid is

$$M_2 - M_1 = \rho h u_1(v_2 - v_1)$$

and the force L on the vane is

$$L = M_1 - M_2 = \rho h u_1(v_1 - v_2) \tag{15}$$

For the direction perpendicular to MN, the change in momentum of the fluid in unit time is

$$M_2' - M_1' = \rho h u_1(u_2 - u_1)$$

But in order to satisfy the condition of continuity,

$$h u_1 = h u_2 \qquad \text{or} \qquad u_1 = u_2$$

and this change in momentum is therefore equal to zero. The force due to differences in pressure on the ends of the area acting in the directions u_1 and u_2 is $h(p_1 - p_2)$. Since the change of momentum in this direction is zero, the total force on the body $ABDC$ in this direction is also zero. Then the pressure force must be equal and opposite to the force exerted by the vane. The force or drag on the vane is

$$D = h(p_1 - p_2)$$

According to Bernoulli's equation,

$$p_1 + \frac{\rho V_1^2}{2} = p_2 + \frac{\rho V_2^2}{2}$$

and, on introducing the components of the resultant velocities and noting that $u_1 = u_2$, this relation may be put in the form

$$p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2)$$

Thus the drag is

$$D = \frac{\rho h}{2}(v_2^2 - v_1^2) \tag{16}$$

In the case of the rotating cylinder, it was found that the circulation around the cylinder was of considerable importance in the determination of the value of the lift force. It is now proposed to determine if such a relationship exists for the lifting vane. If a positive circulation around the area $ABDC$ is taken in the counterclockwise direction, then

$$\Gamma = h(v_1 - v_2) \tag{17}$$

and the substitution of this value in Eqs. (15) and (16) leads to the results

$$L = \rho\Gamma u_1 \tag{18}$$

$$D = -\frac{\rho\Gamma(v_1 + v_2)}{2} \tag{19}$$

In order to determine the values of these force components for a single blade acting independently, the spacing between the blades is increased indefinitely, the blade within the area $ABDC$ being fixed in position. At the same time the end sections AC and BD are moved infinite distances ahead of and behind the vane. During this process the circulation is to be kept constant; it follows at once from Eq. (17) that, if h approaches infinity, then in the limit

$$v_1 - v_2 = 0 \qquad \text{or} \qquad v_1 = v_2$$

But at an infinite distance ahead of the vane the stream is perpendicular to the line MN so that v_1, and therefore v_2, must be equal to zero. After this limiting process is carried out, there remains only a single blade immersed in an infinite stream of fluid and having a circulation Γ around it. If the velocity of the stream at infinity is represented by V_0, then the force components acting on this blade are

$$L = \rho\Gamma V_0 \tag{20}$$

and

$$D = 0 \tag{21}$$

It will be noted that these results are identical with those obtained for the rotating cylinder. Thus, in its general form, the Kutta-Joukowski theorem states that, for an infinitely long cylinder of any cross section having a circulation Γ around it, the lift force per unit of length, measured perpendicular to the undisturbed stream, is given by Eq. (20). As in the case of the rotating cylinder, the circulation around a lifting vane produces an increase in the velocity on the upper surface and a decrease in the velocity on the lower surface as compared with the values that would exist if there were no circulation. The application of Bernoulli's theorem shows at once that there must be a suction on top of the vane and an increase in pressure underneath it. The lift force may be regarded as the resultant of the upward components of these pressures.

The theoretical result that the drag force or resistance to motion is equal to zero is an example of the so-called paradox of D'Alembert and is, of course, contrary to observed facts. This paradox results from the facts that no consideration has been given the effects of viscosity in producing skin-friction forces over the surface of the vane and also that the flow is assumed to be streamline in character throughout and without any formation of wake or eddies behind the body. Actual lifting vanes, such as airplane wings, are of finite span and there is an additional source of drag in the flow that is produced at the tips. This problem of the so-called induced drag will be discussed in Chap. XII.

72. The Development of Circulation.—In the study of the dynamics of the rotating cylinder, it was not difficult to explain how the circulation in the surrounding fluid was produced. The rotational motion of the periphery of the cylinder and the actual viscosity of all real fluids are the conditions responsible for the development of the circulation in this case. In the case of a lifting vane, the production of circulation is a much more complicated process and requires more detailed study. The explanation presented here is essentially that given first by Prandtl.[1]

[1] PRANDTL, L., Applications of Modern Hydrodynamics to Aeronautics, *NACA Tech. Rept.* 116.

In the classical hydrodynamics of nonviscous incompressible fluids, it is demonstrated that, for any curve drawn in a body of fluid so as to consist always of the same fluid particles, the circulation along that curve remains constant with time. On the basis of this statement, which is known as Thomson's theorem, it would appear that, if a body of fluid is initially at rest and is to be made to flow past a lifting vane, then no circulation can be developed because of the fact that its value was initially zero and must remain constant with time. There is a theoretical flow which satisfies these conditions and there are also indications obtained from visual studies of actual flows that in the early stages of its development the flow is of this type, that is, without circulation. However, the theory also shows that, for a lifting vane set at an inclination so that the lift is different from zero, the velocity at the rear edge of the vane tends to become infinite. Such a velocity could not exist in nature; in the actual flow, differences from the theoretical pattern of streamlines are first noted in the neighborhood of the rear edge. It is found that the layers of fluid which pass over the upper and lower surfaces of the vane meet at the trailing edge with slightly different velocities, with the result that at first a so-called surface of discontinuity is formed, across which there is a sudden variation in the magnitude of the velocity. Such a surface is inherently unstable and very quickly rolls up into an eddy or vortex, which is a type of circulatory flow that is accompanied by a circulation. The details of the process by which the surface of discontinuity and the subsequent eddies are formed will be discussed more completely in the chapter that follows.

The eddy generated at the rear edge of the vane increases in strength until the circulation accompanying it reaches a value which produces a finite velocity at the trailing edge of the vane. The flow is then fully developed and steady conditions prevail if no alteration is made in the velocity of the main stream or in the position of the vane. The generation of this so-called starting vortex is accompanied by the production around the wing itself of a countercirculation whose strength is equal but opposite to that produced by the eddy. For a curve surrounding the entire system of wing and starting vortex, the total circulation is equal to zero in accordance with Thomson's theorem. A stage in the development of this flow is shown in Fig. 114.

In the actual flow, the eddy, after reaching the proper strength, breaks away from the airfoil and moves downstream with the general fluid motion, leaving behind only the circulation around the vane. Thus the starting eddy is of significance only in explaining how the circulation around the vane originates; once the circulation is fully developed, only its final value need be considered in computing the lift force acting on the vane.

73. The Lift Coefficient for the Lifting Vane.—The theoretical determination of the value of the circulation around a lifting vane which will lead to the proper adjustment of the velocities at the rear edge of the vane is a problem of too advanced a nature to consider here. The basis of the method involves the use of a

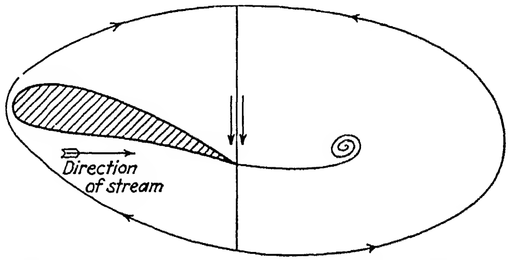


Fig. 114.—Development of circulation around a lifting vane.

conformal transformation of the circular section with its accompanying translatory and circulatory flows into a shape suitable for a lifting vane. The application of this method requires a rather extensive use of the theory of functions of complex variables and only the final results can be presented in this text. For a lifting vane whose chord is l, the theory shows that finite velocities will be maintained at the rear edge if the circulation around the vane has the value

$$\Gamma = \pi l V_0 \sin \alpha_0 \tag{22}$$

In this expression α_0 denotes the inclination or angle of attack of the section referred to the direction of the undisturbed stream and measured from the zero lift axis. This axis is ordinarily drawn as a line through the rear edge of the section parallel to the direction of the fluid stream when the lift force is equal to zero. In practice it is frequently more convenient to measure the angles of attack from the chord line rather than the zero lift

axis, so that, if β represents the angle between these two lines as shown in Fig. 115, then

$$\alpha_0 = \alpha + \beta \tag{23}$$

α being the angle between the chord line and the velocity vector. Thus the circulation may also be expressed in the form

$$\Gamma = \pi l V_0 \sin (\alpha + \beta) \tag{24}$$

The substitution of the values in Eqs. (22) and (24) for Γ in Eq. (20) for L gives

$$L = \pi \rho l V_0{}^2 \sin \alpha_0 \tag{25a}$$

and

$$L = \pi \rho l V_0{}^2 \sin (\alpha + \beta) \tag{25b}$$

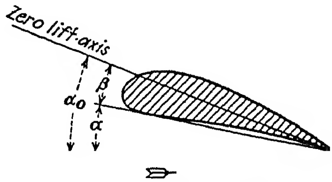


FIG. 115.—Measurement of angle of attack for a lifting vane.

The lift coefficient is now defined as for the rotating cylinder, the area employed being the product of the chord and a unit distance in the direction of the length of the vane. Thus the value of C_L is

$$C_L \quad \frac{}{\rho V_0{}^2} = 2\pi \sin \alpha_0 \tag{26a}$$

or, in terms of α,

$$C_L = 2\pi \sin (\alpha + \beta) \tag{26b}$$

Comparison of the theoretical values of the lift coefficient with those found experimentally shows that the theory is valid only for a small range of angles above and below the zero lift position. For most sections this range varies from about ± 10 to ± 20 deg., so that it is usually permissible to consider the sines of the angles involved in Eqs. (26a) and (26b) as equal to the angles themselves when measured in radians. The lift coefficient may then be expressed in the approximate forms

$$C_L = 2\pi\alpha_0 \tag{27a}$$

and

$$C_L = 2\pi(\alpha + \beta) \tag{27b}$$

It thus appears that for a small range of values, at least the lift coefficient is a linear function of the angle of attack. It is equal to zero when $\alpha = -\beta$. Experimental tests have been carried out in which the condition of two-dimensional flow in planes normal to the length of the vane was maintained. The results[1]

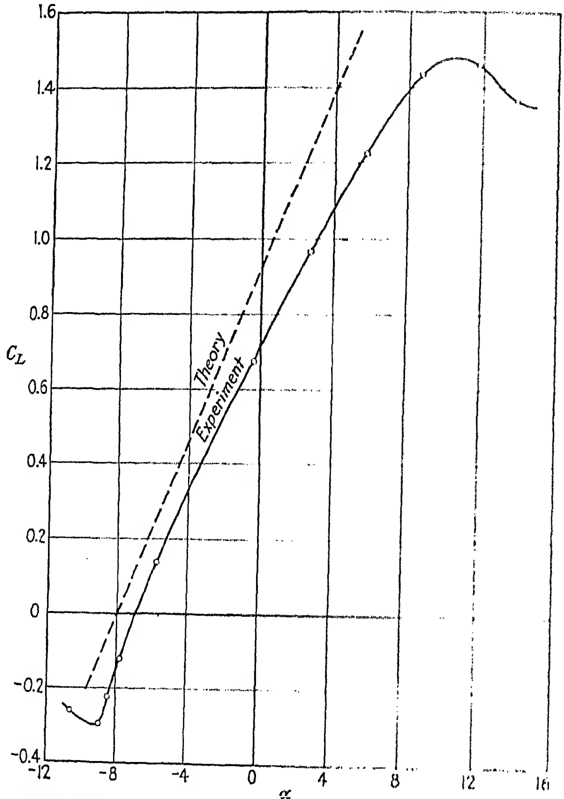


FIG. 116.—Theoretical and experimental curves of C_L versus α for a Joukowski-type lifting vane of infinite length.

for a vane known as the Joukowski type are shown in Fig. 116 The agreement here is much better than that obtained in the case of the rotating cylinder, but it still leaves something to be desired. The two values of the angle of zero lift are reasonably close, while the experimentally determined slope of the curve is

[1] PRANDTL, L., Applications of Modern Hydrodynamics to Aeronautics, *NACA Tech. Rept.* 116.

somewhat less than the theoretical value. The most noticeable difference is found for angles beyond 10 deg., at which the experimental curve shows a maximum while the theoretical one continues on upward as a sine curve [see Eq. (26b)] and would have a maximum value at an angle of 90 degrees. The condition that is responsible for the falling off of the lift in the actual case is known as burbling or stalling of the flow and involves the separation of the fluid from the upper surface of the vane and the formation of a strong eddying wake behind it. Thus the flow no longer is of the streamline character assumed in the theory and as a consequence the results cannot be considered as at all satisfactory beyond the position for which this burbling phenomenon begins. The detailed nature of the mechanism of separation can be explained only by considering the effects of viscosity and its discussion will therefore be postponed to Chap. XII, in which problems of skin friction and resistance in general are treated.

The difference between the theoretical and experimental values of the lift curve slope is due to the fact that the vane does not operate at the full theoretical efficiency in the production of circulation. Experimental data are available which indicate that the lift force is properly given by Eq. (20) if Γ is also determined from experimental measurements of the velocity field around the vane. The expressions for the lift coefficient are more nearly correct when written in the forms

$$C_L = 2\pi k\alpha_0 \tag{28a}$$

and

$$C_L = 2\pi k(\alpha + \beta) \tag{28b}$$

where k is an efficiency factor which for usual vane sections has a value of approximately 0.885. These last equations are, of course, valid only for angles of attack below the stalling position.

Example.—An airplane weighing 4500 lb. has a wing area of 210 sq. ft. and a span of 35 ft. Determine its lift coefficient when flying horizontally at 185 m.p.h. in air of standard density. Compute the theoretical values of the circulation and the angle of attack measured from the zero lift axis.

Solution.—In level flight the weight is equal to the total lift; hence

$$L_T = W = C_L\frac{\rho V_0^2}{2}S$$

from which

$$C_L = \frac{2W}{\rho V_0{}^2 S} = \frac{2 \times 4500}{0.002378(185 \times {}^{88}\!/_{60})^2 \times 210} = 0.245$$

The circulation may be obtained from the Kutta-Joukowski theorem, $L = \rho V_0 \Gamma$, where L is the lift per foot of span. Then for the entire wing of span b

$$L_T = \rho V_0 \Gamma b$$

or

$$\Gamma = \frac{L_T}{\rho V_0 b} = \frac{4500}{0.002378 \times 271 \times 35} = 199 \text{ ft.}^2\text{/sec.}$$

The angle of attack measured from the zero lift axis is obtained from the equation $C_L = 2\pi\alpha_0$ so that

$$\alpha_0 = \frac{C_L}{2\pi} = \frac{0.245}{6.28} = 0.039 \text{ rad.} = 2.24°$$

An alternative method is to use the relation $\Gamma = \pi l V_0 \sin \alpha_0$, where l is the chord. Assuming a rectangular wing planform,

$$l = \frac{S}{b} = \frac{210}{35} = 6 \text{ ft.}$$

and

$$\sin \alpha_0 = \frac{\Gamma}{\pi l V_0} = \frac{199}{3.14 \times 6 \times 271} = 0.039$$

Then

$$\alpha_0 = 0.039 \text{ rad.} = 2.24°$$

Problem 186. Calculate the theoretical lift force per foot of span and the lift coefficient for a lifting vane placed in an airstream with its chord line at an angle of attack of 5 deg. The angle of zero lift is −4 deg., the velocity of the stream is 80 m.p.h. and the chord of the vane is 1 ft.

Fig. 117.

187. The circulation around a lifting vane of 5-ft. chord is 500 ft.²/sec. when placed in an airstream having a velocity of 90 ft. per sec. Determine the theoretical values of the lift per foot of span, the lift coefficient and the angle of attack measured from the zero lift axis.

188. A lifting vane has a symmetrical cross section, as shown in Fig. 117, the axis of symmetry being used as the chord line. What are the values of the theoretical lift coefficient at angles of attack of 0, 5 and 10 deg.?

189. What is the theoretical value of the angle of zero lift for a vane having a lift coefficient of 1.2 at an angle of attack of 10 deg.?

190. The wing of an airplane has an area of 100 sq. ft. and the airplane is flying at 125 m.p.h. in standard air. What is the weight of the airplane if the lift coefficient is 0.54?

191. What is the efficiency factor for a vane if its angle of zero lift is -3 deg. and its lift coefficient is 1.00 at an angle of attack of 8 deg. measured from the chord line?

74. General Characteristics of Blade Screws.—The lifting vane as described in the preceding articles is primarily a device for the production of a force at right angles to its motion. The blade screw in general consists of a number of blades or arms mounted radially around a center or hub and rotating about an axis through the center in a plane perpendicular to the axis. As a result of such rotation, there is usually produced a force in the direction of the axis of the screw known as the thrust. The exact nature of this force will depend on the form of the blade screw and also on whether it is placed in a stream of fluid moving normal to its plane of rotation, whether it is moved in the direction of its axis through a stationary mass of fluid or whether it is merely rotated in a body of fluid at rest. There are then three principal types or classes of blade screws typified by the windmill, the propeller and the fan.

The propeller is caused to rotate by the application of a torque from some suitable power unit. Its major function is to develop an axial thrust by means of which the machine to which it is attached may be made to move. The most familiar applications of the propeller are found in aircraft and water craft.

The windmill type of screw is acted upon by a current of fluid moving in the direction of its axis which causes it to rotate. The rotation of the blades develops a certain torque which may be utilized for the generation of power. This action also results in the development of a thrust in the direction of motion of the fluid stream, but this force does no useful work.

In the case of the fan, its rotation in a body of stationary fluid causes the fluid to move in the direction normal to the plane of rotation. In some cases the fan may be placed in a closed duct so that the fluid approaches the plane of rotation with some initial velocity. The action of the device is then quite similar to that of the propeller, with the exception that in the case of the fan it is usually desired to give the mass of fluid the highest possible velocity, while with the propeller the principal function is the production of the largest possible thrust.

The shape of the blades of a screw depends largely on the purpose to which the screw is to be put, but in general the

blades resemble the airplane propeller and the water turbine runner shown in Fig. 118.

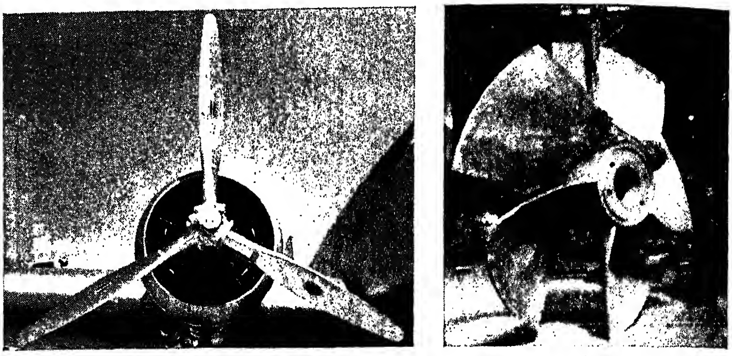


FIG. 118.—Examples of blade-screw forms.

(a) Controllable-pitch airplane propeller. (*Courtesy Hamilton Standard Propellers.*)

(b) Nagler propeller-type water turbine runner. (*Courtesy Allis-Chalmers Manufacturing Company.*)

If the airplane propeller is moving upward, as shown in Fig. 119, with a velocity V_0 and at the same time is rotating about

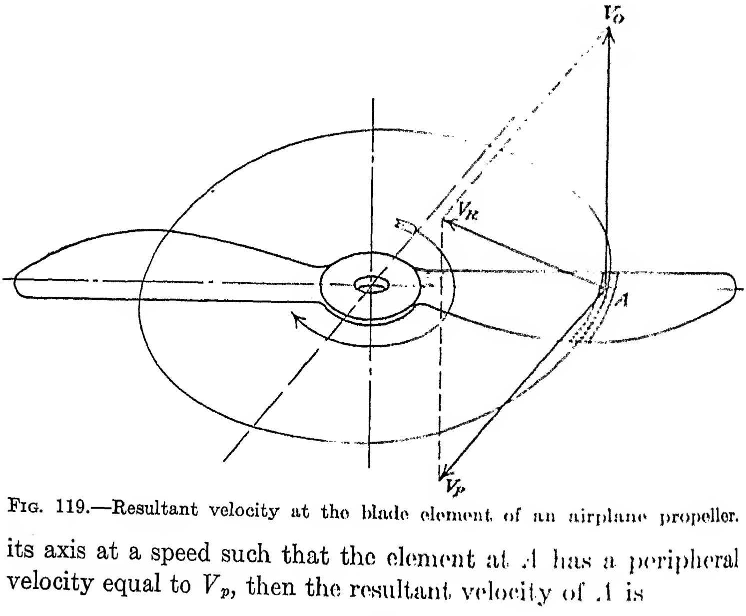


FIG. 119.—Resultant velocity at the blade element of an airplane propeller.

its axis at a speed such that the element at A has a peripheral velocity equal to V_p, then the resultant velocity of A is

$$V_R = \sqrt{V_0^2 + V_p^2} \tag{29}$$

Now the section through the blade of the propeller at the point A is generally constructed in the form of a lifting-vane set so that its chord line is inclined to the plane of rotation at an angle θ, known as the blade angle. This arrangement is shown in Fig. 120, which is a projection in the plane of V_0 and V_p of the blade section and velocity vectors shown obliquely in Fig. 119. The vertical line OB at the right of the diagram represents the axis of rotation, while the horizontal distance OA is equal to the radius $d/2$ of the circle on which the blade element rotates. This distance may also be used to represent the peripheral

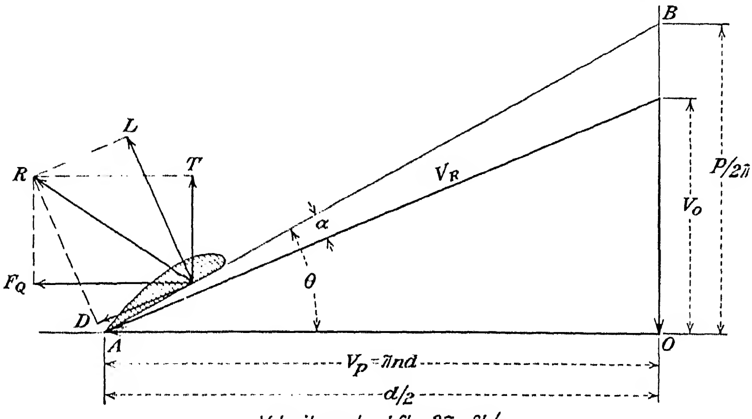


FIG. 120.—Velocity and force components for a propeller-blade element.

velocity $V_p = \pi nd$, in which n represents the rotational speed of the blade in revolutions per second. If V_p is laid off on a scale of 1 ft. = $2\pi n$ ft. per sec., then OA may be considered as the vectorial representation of V_p, its direction being from O toward A. The forward velocity V_0 relative to the blade element is directed vertically downward and may be laid off to the same scale along the axis of rotation with the forward end of the vector located at O. The resultant velocity V_R is the vector sum of V_p and V_0 and is directed as shown in the figure. Thus it appears that the blade section is operating at an angle of attack α whose value is

$$\alpha = \theta - \tan^{-1}\left(\frac{V_0}{V_p}\right) = \theta - \tan^{-1}\left(\frac{V_0}{\pi nd}\right)$$

$$\alpha = \theta - \tan^{-1}\left(\frac{D}{\pi d}\frac{V_0}{nD}\right) \tag{30}$$

where D is the over-all diameter of the propeller. The ratio V_0/n represents the distance that the blade element advances in the time required to make one revolution and is therefore known as the advance per turn. The quantity V_0/nD is called the advance-diameter ratio of the screw.

If the propeller were operating as a mechanical screw in a solid medium, the blade element would advance in one revolution a distance parallel to the axis equal to

$$p = \pi d \tan \theta \tag{31}$$

This distance is known as the geometric pitch of the section. In Fig. 120, the angle BAO is equal to θ, and, if OA is considered as the radius $d/2$, then

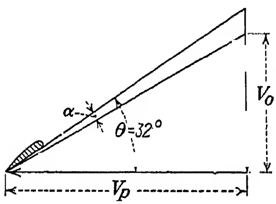
$$\tan \theta = \frac{OB}{d/2}$$

so that

$$OB = \frac{p}{2\pi}$$

In general, the blade angle θ is not constant along the radius, the blades being twisted. The pitch therefore will vary from one section to another, but in practice propellers are specified by the value of their nominal geometric pitch, which is arbitrarily taken as the value of p for the section at two-thirds of the tip radius from the center. The nominal pitch is usually given as a ratio to the propeller diameter.

The advance per turn for a blade screw represents the actual distance that an element moves forward in one revolution and is sometimes called the effective pitch. The value of the effective pitch when the thrust produced by the section is equal to zero is called the experimental pitch. The advance per turn when the thrust of the whole propeller is zero is known as the mean experimental pitch.



Example.—An 8.5-ft.-diameter propeller rotates at 1750 r.p.m. and moves forward at a velocity of 210 m.p.h. Determine the angle of attack of the blade element at 75 per cent of the radius if its blade angle is 32 deg. What are the values of the geometric and effective pitches for this element?

Solution.—The forward velocity is

$$V_0 = 210 \times {}^{88}\!/\!_{60} = 308 \text{ ft./sec.}$$

The peripheral speed at the blade element is

$$V_p = 2\pi rn = \frac{6.28 \times 0.75 \times 4.25 \times 1750}{60} = 584 \text{ ft./sec.}$$

Then, as shown in the figure, the angle between the resultant velocity and the instantaneous plane of rotation is

$$\tan^{-1}\left(\frac{V_p}{V_0}\right) = \tan^{-1}\left(\frac{308}{584}\right) = \tan^{-1}(0.528) = 27°49'$$

The angle of attack is

$$\begin{aligned} \alpha &= \theta - \tan^{-1}\left(\frac{V_0}{V_p}\right) \\ &= 32° - 27°49' = 4°11' \end{aligned}$$

The geometric pitch is

$$p = \pi d \tan \theta = 3.14 \times 0.75 \times 8.5 \times 0.625 = 12.52 \text{ ft.}$$

The effective pitch is

$$\frac{V_0}{n} = 308 \div \frac{1750}{60} = 10.54 \text{ ft.}$$

75. The Blade-element Theory.—It appears from the last article that the elements of a blade screw may be regarded as lifting vanes which move along helical paths. In the case of the propeller, there will consequently be a resultant force produced on the element which will have components of lift and drag respectively perpendicular and parallel to the direction of the resultant motion. These components are shown in Fig. 120. The resultant force R may also be split into components parallel and perpendicular to the axis of rotation. The component T parallel to the axis is the thrust produced by the element, while F_Q, taken at right angles to the axis, is the torque force which must be overcome by the source of power that drives the propeller. Inasmuch as the lift and drag forces acting on the blade element are dependent on the angle of attack α, it is apparent that the thrust and torque force will also be functions of this angle. Furthermore, since α as given by Eq. (30) is a function of the advance-diameter ratio, V_0/nD, it is to be expected that this ratio should be a parameter of fundamental importance in the study of propeller operation. This is true of all types of

blade screws. In studying their performance it is standard practice to consider the variation of such quantities as thrust, torque and efficiency with the advance-diameter ratio as the independent variable.

The analysis of the performance of a blade screw, based on a study of the behavior of the individual elements, regarded as lifting vanes, constitutes the so-called blade-element theory which was first proposed by Drzewiecki.[1] The complete application of this method requires the determination of the thrust and torque components for a number of sections of the blades, and the values for the entire screw are then found by integration along the radius. In some special cases this integration may be carried out analytically, but in the majority of cases graphical methods are employed. In any case the procedure is quite involved and lengthy and will not be taken up in detail here. The reader who is interested in the subject will find full discussions in the literature.[2]

Problem 192. A 9-ft.-diameter propeller rotates at 1800 r.p.m. in an airstream having a velocity of 175 m.p.h. The blade angle of the section at two-thirds of the radius is 30 deg. (*a*) What are the angle of attack and the resultant velocity? (*b*) Determine the values of the elementary thrust and torque force per foot of blade length at this section due to th lift, if the lift coefficient is 0.27 and the chord is 8 in. Neglect the drag fo ce.

193. The ratio of the geometric pitch to the over all di meter for the blade element of a 7.5-ft.-diameter propeller at one-half th radius is 0.6. If the effective pitch is 3.23 ft., what are the angle of attack a l the advance-diameter ratio?

194. A 12-ft.-diameter propeller has the blade sections set at a constant angle of attack of 2 deg. when the forward speed is 120 m.p.h. and the propeller is rotating at 1500 r.p.m. Determine the blade angles of the sections at 25, 50, 75 and 90 per cent of the radius.

76. The Momentum Theory of Propellers. The earliest application of the screw-type propeller is found in the case of seagoing vessels. The first attempts to develop a propeller theory were therefore made by men working in this field, the originator of the theory being W. J. M. Rankine. Extensions of

[1] Pronounced *Je-vee-yet'ski*.

[2] Weick, F. E., "Aircraft Propeller Design," McGraw-Hill Book Company, Inc., New York, 1930.

Glauert, H., "Airplane Propellers," Div. L of "Aerodynamic Theory," edited by W. F. Durand, Julius Springer, Berlin, 1935.

Rankine's work were made by R. E. Froude. The basic concept of the Rankine-Froude theory is that, as the fluid passes through the disk swept out by the propeller blades, its relative velocity with respect to this disk is made greater than the velocity with which the propeller is advancing. The body of fluid that has passed through the propeller disk is known as the slipstream and, as a result of the increased momentum of this slipstream, a thrust is produced on the propeller.

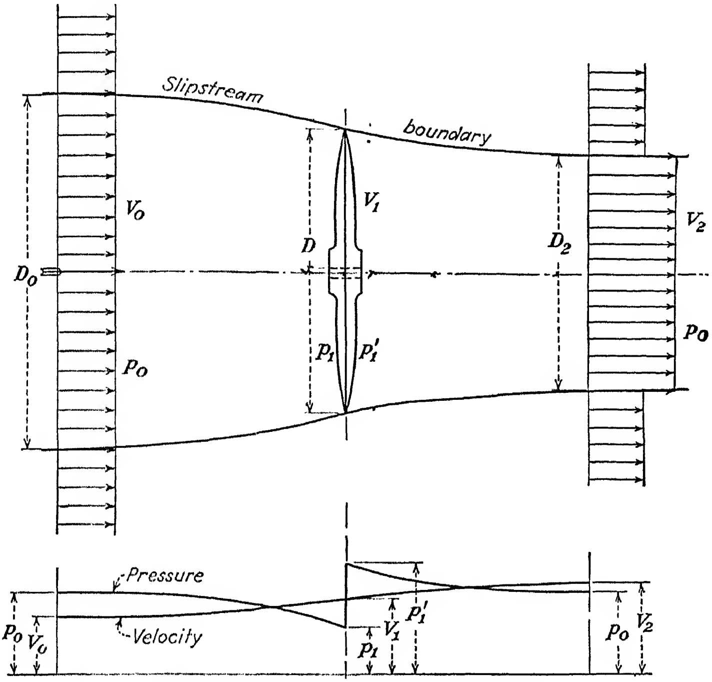


FIG. 121.—The slipstream of a propeller.

In setting up the propeller problem so as to apply the momentum theorem to it, it is convenient to make use of the principle of relative motion and to consider the propeller as rotating at a fixed position in space while a stream of fluid moves past it, as shown in Fig. 121. Far ahead of the propeller there is a current of fluid of infinite extent moving in the direction of the propeller axis with a uniform velocity V_0. As the fluid approaches the propeller disk, the velocity of the portion of the stream directly in front of the disk is increased to a value V_1. At the propeller,

this body of fluid, which constitutes the slipstream, has the same diameter D as the propeller, while far upstream, where the velocity is V_0, its diameter is larger and is represented by D_0. The fluid, after passing through the propeller, experiences a further increase in velocity to the value V_2, with an accompanying reduction in slipstream diameter to D_2. The boundaries of the slipstream are the heavy curved lines shown in Fig. 121.

In the simple momentum theory, the propeller is treated as an actuator disk which is responsible for the change in velocity of the slipstream. Accompanying this change in velocity, there is also a variation in pressure. Far ahead of and behind the propeller, the pressure is that of the undisturbed fluid, represented by p_0. As the actuator disk is approached, the pressure drops to p_1 in accordance with Bernoulli's theorem, but in passing through the disk the pressure experiences a sudden increase. Bernoulli's theorem cannot be applied directly from one side of the propeller to the other because of the fact that energy has been added to the fluid at the actuator disk and the total head of the fluid is therefore increased. On the downstream side of the propeller the pressure is therefore p_1' and from this value it decreases to p_0 at a great distance away. The variations in velocity and pressure along the axis of the slipstream are indicated qualitatively in Fig. 121.

The mass of air passing through any section of the slipstream in unit time may be computed on the basis of conditions at the propeller. This mass is

$$m = \frac{\pi D^2}{4} \rho V_1$$

The flow through the slipstream is assumed to be continuous so that initially this same mass had a velocity V_0, while far behind the propeller it has acquired a velocity V_2. The change in momentum in unit time is therefore equal to

$$\Delta M = m(V_2 - V_0)$$

This change in momentum is equal to the force acting on the propeller in the direction opposite to that of the flow; if this force or thrust is denoted by T, the result is

$$T = \frac{\pi D^2}{4} \rho V_1 (V_2 - V_0) \tag{32}$$

It is now necessary to find some relation between the velocities involved in Eq. (32). The thrust may also be calculated by considering it as equal to the force produced by the difference in pressure on the two sides of the actuator disk. When Bernoulli's theorem is applied to the flow to the left of the propeller, the result is

$$p_0 + \frac{\rho V_0^2}{2} = p_1 + \frac{\rho V_1^2}{2}$$

while for the downstream side

$$p_0 + \frac{\rho V_2^2}{2} = p_1' + \frac{\rho V_1^2}{2}$$

The value of the desired pressure difference may be found by subtracting the first of these equations from the second. This value is

$$p_1' - p_1 = \frac{\rho}{2}(V_2^2 - V_0^2)$$

Hence the thrust of the propeller is equal to

$$T = \frac{\pi D^2}{8}\rho(V_2^2 - V_0^2) \tag{33}$$

A comparison of the values of T given by Eqs. (32) and (33) indicates that the velocities must be related so that

$$V_1 = \frac{V_2 + V_0}{2} \tag{34}$$

Thus the slipstream velocity at the propeller is the arithmetic mean of the velocities far ahead of it and behind it.

Then the total increase in velocity through the slipstream is divided so that half of this increase takes place ahead of the propeller and the remaining half behind it.

The energy absorbed by the propeller may be calculated in either of two ways. This power may be considered as equal to the work done by the propeller on the fluid in unit time and is therefore

$$E = TV_1$$

If the value of T as given by Eq. (33) is introduced, the expression for this energy becomes

$$E = \frac{\pi D^2}{8}\rho V_1(V_2^2 - V_0^2)$$

Since the mass of fluid passing through the slipstream is

$$m = \frac{\pi D^2}{4}\rho V_1$$

it is seen at once that the above expression is equivalent to the change of kinetic energy from one end of the slipstream to the other.

The theoretical efficiency of the propeller may now be expressed as the ratio between the work done by the thrust of the propeller in advancing with a velocity V_0 and the energy absorbed by it, that is,

$$\eta = \frac{TV_0}{TV_1} = \frac{V_0}{V_1} = \frac{2V_0}{V_2 + V_0}$$

If the total increase in slipstream velocity, $V_2 - V_0$, is represented by ΔV, then the velocity at the propeller is

$$V_1 = V_0 + \frac{\Delta V}{2}$$

so that the efficiency may also be expressed in the form

$$\eta = \frac{V_0}{V_0 + \frac{\Delta V}{2}} \tag{35}$$

The thrust as given by Eq. (33) may be written in the form

$$T = \frac{\pi D^2}{8}\rho(V_2 - V_0)(V_2 + V_0)$$

The first term in parentheses is the total increase in slipstream velocity, ΔV, while the second term may be written as

$$V_2 + V_0 = (V_2 - V_0) + 2V_0 = 2V_0 + \Delta V = 2\left(V_0 + \frac{\Delta V}{2}\right)$$

Substituting these quantities in the above expression for T, the result is

$$T = \frac{\pi D^2}{4} \rho \, \Delta V \left(V_0 + \frac{\Delta V}{2} \right)$$

This last expression may be considered as a quadratic equation in ΔV. The value of ΔV in terms of the thrust is found to be

$$\Delta V = -V_0 + \sqrt{V_0^2 + \frac{8T}{\pi D^2 \rho}}$$

Only the positive root of the radical is of any physical significance.

It is now convenient to introduce a thrust coefficient into this expression. There are a number of different forms employed but the one most commonly used in the United States and England is defined by the equation

$$T = C_T \rho n^2 D^4 \tag{36}$$

in which n is the rotational speed in revolutions per second and C_T is a nondimensional coefficient. The value of the velocity increase in terms of C_T is

$$\Delta V = -V_0 \left[1 - \sqrt{1 + \frac{8C_T}{\pi} \left(\frac{nD}{V_0} \right)^2} \right]$$

so that the relationship between efficiency and thrust coefficient finally is

$$\eta = \frac{2}{1 + \sqrt{1 + \dfrac{8C_T}{\pi} \left(\dfrac{nD}{V_0} \right)^2}} \tag{37}$$

The power absorbed by the propeller may also be conveniently expressed in terms of a power coefficient such that

$$E = C_E \rho n^3 D^5 \tag{38}$$

so that an alternative formula for the efficiency is

$$\eta = \frac{TV_0}{E} = \frac{C_T}{C_E} \left(\frac{V_0}{nD} \right) \tag{39}$$

Problem 195. A propeller 5 ft. in diameter moves through water at 20 m.p.h. If it develops a thrust of 3200 lb., what is the total increase in the

relative velocity of the slipstream? What is the relative velocity at the propeller?

196. An 8.5-ft.-diameter airplane propeller develops a thrust of 1200 lb. when flying at 110 m.p.h. and at a rotational speed of 1500 r.p.m. Compute the values of the thrust coefficient, the ideal efficiency and the advance-diameter ratio. What is the theoretical value of the power absorbed by the propeller?

77. Comparison of the Momentum Theory with Experimental Data.—It is now of considerable interest to make a comparison

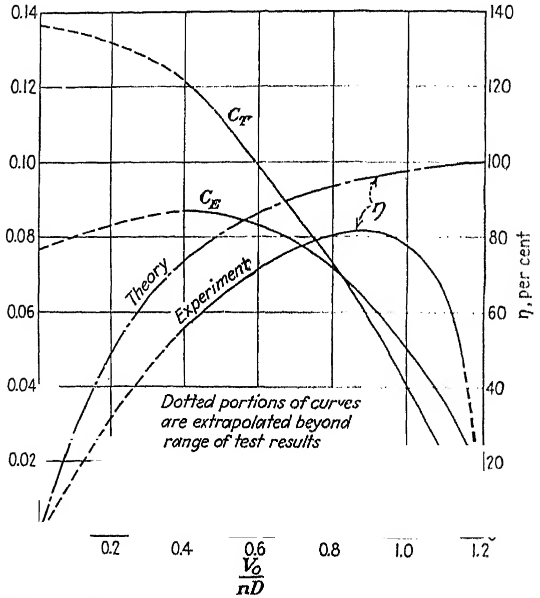


FIG. 122.—Typical performance curves for a model propeller. (*Durand, W. F., Tests of Thirteen Navy Type Model Propellers, NACA Tech. Rept.* 237.)

of the results derived from the above analysis of propeller performance with those obtained experimentally. Tests on propellers are carried out by having the propeller mounted on a test stand so that the torque and thrust developed by the entire unit may be measured at various values of the advance-diameter ratio. A typical set of test results for a model airplane propeller having a pitch-diameter ratio of 1.0 are shown in Fig. 122, in which the values of C_T, C_E and η have been plotted against V_0/nD. The theoretical values of the efficiency, based on Eq. (37) and using the experimentally determined thrust coefficients,

are also shown. The theoretical efficiency curve is seen at once to be considerably higher than the experimental one, owing mainly to the fact that the theory is based on a number of simplifying assumptions. It has been assumed that the propeller blades can be rotated without any frictional losses, while such questions as rotational flow and interference have also been set aside. Thus, while the theoretical efficiency approaches a unit value when the thrust coefficient approaches zero, the actual efficiency drops to zero at this point because of the fact that the power coefficient is still finite. In spite of the large discrepancies between the theoretical and actual efficiency curves, the theory is, nevertheless, of considerable use in that the theoretical efficiency represents an ideal that an actual propeller may only partially approach.

A more detailed study of the propeller would involve considerations of the effect on performance of such characteristics as blade planform, shapes of vane sections, variations of the nominal pitch and the variation of pitch along the radius. The operating characteristics of fans and windmills may also be studied by applications of the momentum theorem similar to that used for the propeller, but it is not possible to consider these problems in detail here.

General Problems

197. A circular cylinder 8 ft. in diameter is mounted with its axis vertical on a flat car and is rotated at 225 r.p.m. The car moves along a straight track at 30 m.p.h. into a head wind of 20 m.p.h. which is directed at an angle of 30 deg. to the track. What is the lift force per foot of length of the cylinder and what is its component along the track?

198. Calculate the circulation and theoretical lift per foot of length for a cylinder 3 ft. in diameter if the velocity at the point $\theta = 45$ deg. (see Fig. 109) on its surface is 30 ft. per sec. The cylinder is rotating in an airstream having a velocity of 20 ft. per sec.

199. The landing speed of an airplane, which is determined by the maximum lift coefficient of its wings, is to be 75 m.p.h. Assume that the airplane is flying at sea level in standard air. (*a*) If the wings carry an average load of 20 lb. per sq. ft., what is the maximum value of the lift coefficient? (*b*) If the airplane weighs 4200 lb. and the maximum lift coefficient of the wings is 1.3, what is the wing area?

200. A plank having an area of 20 sq. ft. is towed under water at a velocity of 25 m.p.h. If its lift coefficient is 0.64, what load will it support, neglecting buoyancy?

201. A kite is made in the form of a plane surface having an area of 12 sq. ft. It is flown in a horizontal wind current having a velocity of 15 m.p.h.

so that its surface makes an angle of 6 deg. with the horizontal as shown in Fig. 123. If the string is at an angle of 45 deg. with the horizontal and has a pull of 2 lb., what is the weight of the kite, assuming that it develops its full theoretical lift?

202. The blade element at the three-quarters radius section on an 8-ft.-diameter propeller has a blade angle of 19 deg. The lift and drag forces on this element are 26.7 lb. per ft. and 1.78 lb. per ft., respectively. The forward velocity is 120 ft. per sec. and the rotational speed is 1250 r.p.m. Determine the values of the thrust and torque force per foot of blade length acting on this element and the angle of attack of the element.

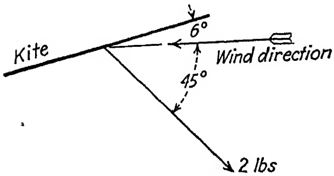


FIG. 123.

203. A 10-ft.-diameter propeller has the blade element at two-thirds of the radius set at a blade angle of 25 deg. Determine the value of its angle of attack for rotational speeds of 1000, 2000 and 3000 r.p.m. when the forward velocity is 120 m.p.h. If the blade section develops its maximum lift coefficient at an angle of attack of 12 deg., what should be the limiting value of the rotational speed of this propeller?

204. A 7-ft. propeller is mounted on a test stand in an airstream having a velocity of 90 m.p.h. At a rotational speed of 1200 r.p.m. the measured values of the thrust and power input are 750 lb. and 240 hp., respectively. Determine the values of thrust and power coefficients and the ideal and actual efficiencies.

CHAPTER VIII

THE FLOW OF VISCOUS FLUIDS

78. Effect of Viscosity.—In the foregoing pages the discussions have dealt with ideal or perfect fluids; gases have been assumed to be without viscosity and perfectly elastic; liquids have been taken as nonviscous and incompressible. In hydrostatics these assumptions do not introduce any error except in the case of liquids at extremely high pressure. In dealing with flowing fluid, however, an explanation of many of the phenomena requires an adequate conception of viscosity and its effect. This is true not only when the flowing fluid is oil, as in lubrication problems, or some other of the many viscous materials found in engineering and industry, but also in dealing with air and water, the commonest of flowing fluids. Even though the latter are of relatively low viscosity, this property is often the prime factor in determining the quantity or character of their flow. Viscosity is in fact the greatest single difference between ideal and real fluids and for that reason this chapter is devoted largely to a consideration of its nature and effect on fluid flow.

79. Reynolds' Experiment on Flow in Pipes.—In the earlier discussion of nonviscous fluids, the flow was classified as steady motion or unsteady motion, having in mind the flow of the stream as a whole. The same classification can be made in the case of viscous fluids. It is important to note, however, that, even if the motion of a stream as a whole is steady, conditions at points within the stream may be quite unsteady and the detailed structure is very complicated. The nature of this deviation from steady flow within the stream is defined most clearly by describing the classical experiments of Osborne Reynolds,[1] an English scientist, who was the first to demonstrate its existence.

[1] Reynolds, O., An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Will Be Direct or Sinuous, and of the Law of Resistance in Parallel Channels, *Phil. Trans. Roy. Soc., London*, 1883, or *Sci. Papers*, vol. 2, p. 11.

Reynolds' experiments were made with an apparatus such as that shown in Fig. 124. A straight piece of round glass tube with a flared inlet is placed in a glass-walled tank full of water at rest. One end extends through the wall of the tank and is fitted with a valve that controls the rate of flow. A small reservoir of colored liquid is arranged so that it discharges the dye through a fine nozzle into the inlet end of the tube with the same velocity as the water, and the behavior of this jet of dye indicates qualitatively the type of flow in the pipe.

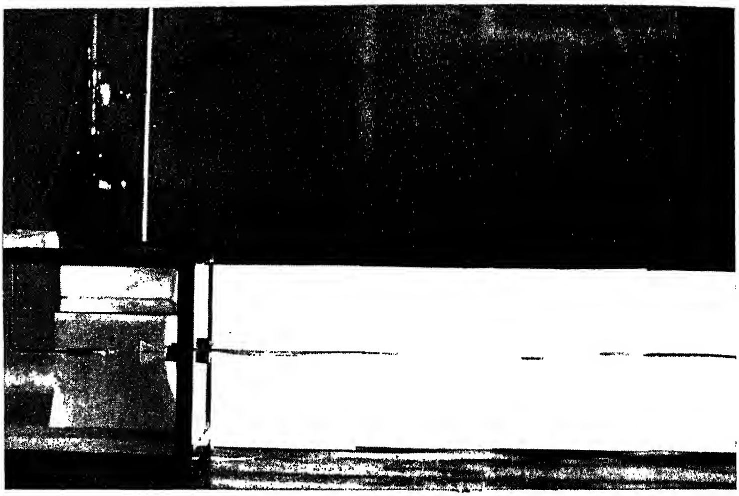


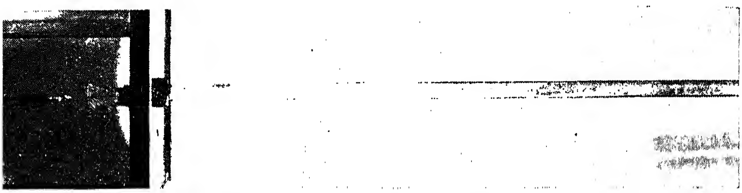
FIG. 124.—Apparatus for studying pipe flow.

When the velocity in the tube was maintained at a sufficiently low value, the jet of dye traveled down the pipe as a straight line without appreciable disintegration as shown in Fig. 125*a*, indicating the existence of what Reynolds termed "direct" or "streamline" flow. This type of motion was steady in character. The next step in the experiments was to increase the velocity of flow until a speed was reached at which the jet of dye no longer traversed the length of the pipe as an unbroken line, but mixed more or less completely with the surrounding water, indicating a condition of flow which Reynolds called "sinuous." In the latter case, shown in Fig. 125*b*, the motion as a whole could still be regarded as steady in character if the pressure in the pipe and the rate of discharge were maintained as constant, but in its detailed structure the flow was unsteady.

80. Laminar and Turbulent Flow.—The most commonly used modern terms for the two kinds of motion seen in Reynolds' experiment are laminar and turbulent and these expressions will be employed throughout the remainder of this work. Laminar flow is also described as viscous or nonsinuous motion. These two types of motion also characterize many cases of flow other than that in pipes, for example, the flow near the surface of an airfoil or sphere.

In laminar flow the fluid moves in parallel laminas or layers, the velocities in these laminas not necessarily being the same, and

(*a*) Streamline or laminar flow.



(*b*) Sinuous or turbulent flow.
FIG. 125.—Types of flow in pipes.

at any point in the fluid the velocity is independent of the time. In turbulent flow the velocity at any point may be varying both in magnitude and in direction with time, while the average velocity may vary from point to point just as in laminar flow. Thus, in the case of a two-dimensional or plane flow which is turbulent, the velocity components at a particular point might be represented by expressions of the form

$$u = \bar{u} + u' \qquad = \bar{v} + v' \tag{1}$$

where $\bar{u}$ and $\bar{v}$ are the time averages of these velocity components and in general are functions of the coordinates of the point, while u' and v' are the deviations from these average values and are functions of the time as well as the coordinates. Ordinarily the deviations are rather irregular functions of time.

As will be shown later, there is a fundamental criterion which determines whether a fluid motion is of the laminar or of the turbulent type. In the case of Reynolds' experiments, with water at a given temperature and for a pipe of a given diameter, the average velocity of the flow is the determining factor. The shape of the entrance to the pipe and the condition of the water in the tank before the valve is opened are other factors of great importance. Although considerable information can be obtained about the nature and values of this criterion both by theoretical and by experimental methods, there is at present very little knowledge of the actual mechanism that is involved when a flow changes from laminar to turbulent. The situation is somewhat analogous to that which existed in the science of electricity a decade or so ago, before the developments of modern physics, in that it is at present possible to calculate the effects of this flow phenomenon although there is no completely satisfactory theory as to its inherent character.

81. Basic Hypotheses Concerning Viscosity.—In considering the effect of viscosity on fluid flow, it is first necessary to introduce two fundamental assumptions. These assumptions form a basis for certain theories and will be justified by a comparison of such theories with experimental results. The first of these assumptions is that, wherever a fluid is in contact with a solid boundary or wall, there is no motion or slip, relative to the boundary, of the fluid particles immediately adjacent to it. In other words, the fluid adheres or sticks to the boundary surface. The second assumption was first introduced by Newton. His hypothesis states that the shearing stress between adjacent layers of fluid of infinitesimal thickness is proportional to the rate of shear in the direction perpendicular to the motion. If a particle of fluid, shown in Fig. 126a with sides of lengths Δx and Δy, is set in motion with a velocity u on its lower surface and $u + \Delta u$

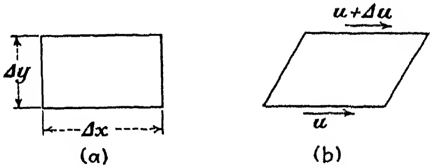


FIG. 126.—Motion of a viscous fluid particle.

on the upper, it will be distorted into the form shown in Fig. 126b. The rate of shear in the direction normal to the motion is then $\Delta u/\Delta y$ if the particle is assumed to be moving in the x-direction. In a general case the velocity u will vary with both coordinates x and y for points throughout the fluid, so that, if the particle is infinitesimally small, the limiting value of the rate of shear at any point is $\partial u/\partial y$, the partial derivative of u with respect to y. This expression is often called the velocity gradient. Newton's assumption in regard to the value of the shearing stress τ at any point in the fluid may now be written as

$$\tau = \mu\frac{\partial u}{\partial y} \tag{2}$$

where μ is a coefficient of proportionality.

An analogy is often drawn between the coefficient μ and the shear modulus of elastic materials. The latter is the ratio of shear stress to unit deformation while μ may be written as $\dfrac{\tau}{\partial u/\partial y}$, the ratio of shear stress to transverse velocity gradient. There is, however, an important distinction between the effects of shear stress on solids and on liquids, in that a given stress on a solid produces a definite deformation while a given stress on a fluid produces continuous deformation at a definite rate.

82. Definition of Viscosity.—In the study of the mechanics of elastic solids, the coefficient of proportionality between stress and strain is known as the modulus of elasticity. A similar proportionality factor in the mechanics of viscous fluids is represented by the coefficient μ which appears in Eq. (2) and is known as the coefficient of viscosity or, more simply, the viscosity. The nature of this coefficient may be indicated more clearly by considering the laminar motion of a viscous fluid bounded by two flat parallel plates, one of which is stationary and the other moving parallel to its surface with a constant velocity V. The plates are assumed to be very large so that the flow may be considered as two-dimensional; a cross section of it made by a plane perpendicular to the plates and parallel to the direction of motion is shown in Fig. 127. If the upper plate is the one which is in motion with a velocity V and if the velocity of the fluid is referred to axes fixed with respect to the stationary plate, then, according to the fundamental hypothesis of no slip

at the boundaries, the fluid must have zero velocity at the lower plate and a velocity equal to V at the upper. The particles in the uppermost layer are carried along with the moving plate and this layer in turn imparts a forward motion to the one immediately below it, this effect being transmitted downward through the fluid with an intensity that diminishes as the dis-

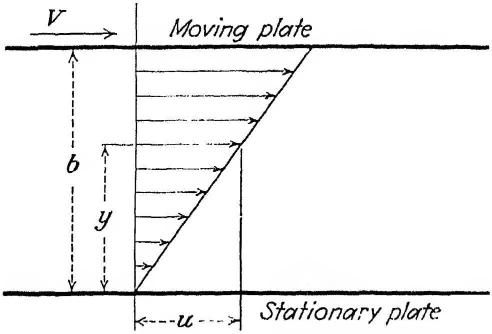


FIG. 127.—Laminar flow between parallel plates in relative motion.

tance from the upper plate is increased. Thus at any point at a distance y above the lower plate the velocity is

$$u = \frac{Vy}{b} \tag{3}$$

where b is the distance between the plates. The rate of shear is $\dfrac{\partial u}{\partial y} = \dfrac{V}{b}$ and in this case is constant throughout the fluid. Then the shearing stress at any point is

$$\tau = \mu \frac{V}{b} \tag{4}$$

The force necessary to move the plate against the resistance produced by the motion of the fluid is simply the value of τ multiplied by the area of the plate. A force of the same magnitude, but acting in the opposite direction, must also be applied to the stationary plate to hold it in a fixed position.

This arrangement of two parallel plates separated by fluid serves as the basis for the standard definition of the coefficient of viscosity as first proposed by Maxwell. If Eq. (4) is solved for μ, the result is

$$\mu = \tau \frac{b}{V} \tag{5}$$

If the velocity of the moving plate and the gap between the plates are both taken as unit values, then μ is exactly equal in magnitude to the force per unit area acting on either plate. Maxwell's definition of viscosity stated in words is as follows:

If two horizontal plane surfaces are placed parallel to one another and at a unit distance apart, the space between them being completely filled with fluid, and if one plate is moved in a direction parallel to its surface with a unit velocity relative to the other, then the force per unit area acting on either plate in the form of a resistance to the motion is numerically equal to the viscosity or coefficient of viscosity of the fluid between the plates.

In dealing with viscosity mathematically it is convenient to take μ from Eq. (2), that is,

$$\mu = \frac{\tau}{\partial u/\partial y} \tag{6}$$

and treat it as the ratio of shear stress to transverse velocity gradient. Maxwell's definition merely refers to a special case of this more general formula.

83. Kinematic Viscosity and Fluidity.—In some problems it is convenient to introduce a modified form of the coefficient of viscosity which is obtained by dividing the value of μ by the density of the fluid, ρ. This quantity, which is usually denoted by the symbol ν, has the value

$$\nu = \frac{\mu}{\rho} \tag{7}$$

and is known as the kinematic coefficient of viscosity, as distinguished from the coefficient μ, which is frequently called the absolute viscosity.

Fluidity is a term used to indicate the facility with which a fluid flows. Qualitatively this property is the antithesis of viscosity, and quantitatively it is the reciprocal of the absolute viscosity. Then the fluidity is

$$\varphi = \frac{1}{\mu} \tag{8}$$

The specific viscosity of a fluid is defined as the ratio of its absolute viscosity to that of water at 20°C. Laboratory and commercial methods for the determination of these various coefficients will be studied in Chap. XVI, while numerical values of μ and ν for some of the commoner fluids will be given in Art. 88.

84. Dimensions and Units of Viscosity.—The coefficients of viscosity as defined by the statements given in the two preceding articles are not dimensionless quantities since their values will depend on the units employed for their measurement. On introducing the fundamental units of mass, length and time in Eq. (5),

$$\mu = \tau \frac{b}{V} \approx \frac{\text{force}}{\text{area}} \frac{\text{length}}{\text{velocity}} \approx \left(\frac{ML}{T^2} \cdot \frac{1}{L^2}\right)\left(\frac{L}{L/T}\right) \approx \frac{M}{LT} \tag{9}$$

from which it appears that μ has the dimensions M/LT. The coefficient of viscosity may thus be measured in slugs divided by foot-seconds in English units, or in grams mass divided by centimeter-seconds in the metric system. This latter combination of units is called a poise in honor of Poiseuille, one of the earliest experimenters in the field of viscous fluid motion. The poise may be divided into 100 equal parts, giving the unit known as the centipoise. It happens that the viscosity of water at 20°C. is almost exactly 1 centipoise so that the viscosity in centipoises is numerically equal to the specific viscosity as defined in the last article.

The kinematic coefficient of viscosity as defined by Eq. (7) is readily seen to have the dimension L^2/T, that is, feet squared per second or centimeters squared per second. The unit of kinematic viscosity in the metric system is known as the stoke in honor of Sir George Stokes, an English mathematician who contributed much to the theory of viscous fluids. A one-hundredth part of a stoke is called a centistoke.

The problem of conversion of the units of viscosity from one system of measurement to the other requires some detailed attention. Confusion sometimes results because of the fact that most data on viscosity are found in tables prepared by chemists and physicists, in which an absolute system of units is employed, while in engineering work the gravitational system is commonly used. For comparison, the absolute and kinematic

viscosities of air will now be expressed in both metric and English units.

The viscosity of air at 15°C. and a pressure of 1 atmosphere as given in the International Critical Tables[1] is

$$\mu = 1.783 \times 10^{-4} \frac{\text{g.}}{\text{cm.sec.}} \text{ or poises} \tag{10}$$

Since the dimensions of μ are M/LT, it is apparent that the gram used in Eq. (10) is the gram mass. This unit is defined as the mass of 1 cc. of water at 4°C. and a pressure of 1 atmosphere. In order to find the kinematic viscosity of air, its density must be determined in the same units as those used in Eq. (10). The specific gravity of air is

$$\text{Specific gravity} = \frac{0.0765}{62.42} = 1.225 \times 10^{-3}$$

and, since the density and specific gravity are numerically equal, the density of air at 15°C. is

$$\rho = 1.225 \times 10^{-3} \quad \text{g./cm.}^3$$

The kinematic viscosity of air is then

$$\begin{aligned} \nu = \frac{\mu}{\rho} &= \frac{1.783 \times 10^{-4}}{1.225 \times 10^{-3}} \frac{\text{g./cm.sec.}}{\text{g./cm.}^3} \\ &= 1.456 \times 10^{-1} \text{ cm.}^2/\text{sec.} \end{aligned}$$

This value may be converted to the English system of measurement by changing centimeters to feet. Then

$$\nu = \frac{1.456 \times 10^{-1}}{(30.48)^2} \text{ ft.}^2/\text{sec.} = 1.567 \times 10^{-4} \text{ ft.}^2/\text{sec.}$$

The absolute viscosity μ may be expressed in the English system by multiplying the above value of ν by the density expressed in the proper units. The density of air is

$$\rho = 0.002378 \text{ slugs/ft.}^3,$$

so that

$$\begin{aligned} \mu &= 1.567 \times 2.378 \times 10^{-7} \text{ (ft.}^2/\text{sec.) (slugs/ft.}^3) \\ &= 3.726 \times 10^{-7} \text{ slugs/ft.sec.} \end{aligned} \tag{11}$$

[1] "International Critical Tables," vol. V, pp. 2–3, McGraw-Hill Book Company, Inc., New York, 1926–1930.

The value of μ in English units may also be obtained directly from Eq. (10) if the proper units of mass are employed. In order to carry out this conversion, it is necessary to determine the relation between the gram, which is the unit of mass in the metric system, and the slug used in the English system. In the absolute metric system the unit of mass is the gram and the unit of force is the dyne, which is that force which will give a mass of 1 g. an acceleration of 1 cm. per sec. per sec. The weight of a gram mass or the force produced on it by gravitational attraction is therefore 981 dynes. In the gravitational system this force is regarded as the unit of weight and is known as the gram weight. Hence

$$1 \text{ g.wt.} = 981 \text{ dynes} = 981 \text{ g.cm./sec.}^2$$

or

$$1 \text{ g.} = \frac{1}{981} \frac{\text{g.wt.}}{\text{cm./sec.}^2}$$

But

$$1 \text{ g.wt.} = 0.002205 \text{ lb.}$$

and

$$1 \text{ lb.} = 1 \text{ slug ft./sec.}^2$$

Also noting that 981 cm./sec.^{2} = 32.2 ft./sec.^{2}, the equivalent of the gram mass is finally found to be

$$\begin{aligned} 1 \text{ g.} &= \frac{0.002205 \text{ slugs ft./sec.}^2}{32.2 \text{ ft./sec.}^2} \\ &= 6.85 \times 10^{-5} \text{ slugs} \end{aligned}$$

If this relationship is introduced into the value of μ as given by Eq. (10), and the unit of length in centimeters occurring therein is changed to feet, then

$$\begin{aligned} \mu &= \frac{1.783 \times 6.85 \times 10^{-9}}{1/30.48} \frac{\text{slugs}}{\text{ft.sec.}} \\ &= 3.723 \times 10^{-7} \frac{\text{slugs}}{\text{ft.sec.}} \end{aligned}$$

which agrees closely with the result given by Eq. (11).

In comparing the two systems of units, it should be noted that in the English gravitational system of measurement there is available the unit of mass known as the slug, which is equivalent to a 1-lb. force divided by a unit acceleration, while in the

absolute metric system the unit employed is the gram mass, which is equal to a gram weight divided by the acceleration of gravity.

Problem 205. The space between two parallel horizontal plates which are ½ in. apart is filled with oil having an absolute viscosity of 0.032 slugs per ft. sec. If the upper plate is moved with a velocity of 10 ft. per sec. and the lower one is stationary, what is the shear stress in the oil?

206. The absolute viscosity of a fluid having a specific gravity of 0.8 is 0.890 poises. Determine the kinematic viscosity in stokes and the fluidity.

207. The absolute viscosity of water at 15°C. is 0.01144 poises. Compute the kinematic viscosity in stokes and the kinematic and absolute viscosities in the English system.

208. The kinematic viscosity of an oil is 0.017 ft.^{2} per sec. and the specific gravity is 0.85. Determine the absolute viscosity in the English and metric systems and the kinematic viscosity in the metric system.

85. Laminar Flow in Circular Pipes. The Hagen-Poiseuille Law.—One of the earliest important studies in the field of viscous flow was an experimental investigation of the characteristics of laminar flow in straight pipes of circular cross section. This work was done independently by two men, the first being Hagen, a German engineer whose results were published in 1839, while the second investigation was that of Poiseuille, a French scientist, whose first work on the subject was issued in 1840. Hagen experimented with water flowing through brass tubes, while Poiseuille worked with water flowing through fine capillary tubes, since he was interested in the behavior of blood as it flows through the veins of the body.

As a result of the studies of Hagen and Poiseuille it was determined that the quantity of a given viscous liquid which flows through a small tube in a given time is proportional to the pressure difference causing flow, to the fourth power of the diameter of the tube, and inversely to its length. This is known as the Hagen-Poiseuille law.

This law was later derived theoretically, and will be proved here by considering the equilibrium of a body of fluid moving through a pipe. In Fig. 128 are shown the longitudinal and cross sections of a horizontal straight circular pipe of internal diameter d. The flow is laminar in character and is assumed to be steady; attention is to be focused on the cylindrical portion of fluid, $MNOP$, of length l and diameter $2y$. If the fluid is moving from left to right through the pipe, there being an average pressure p_1 on the left end and p_2 on the right, then the force on

the cylinder in the direction of motion produced by this pressure difference is

$$(p_1 - p_2)\pi y^2$$

If it is assumed that the flow is steady and that the velocity does not change along any line parallel to the axis of the pipe, then there is no accelerating force acting on any of the fluid particles. There is, however, a force due to the shearing stresses acting on the outer surface of the cylinder of fluid. Since the velocity is independent of x, the distance measured along the axis of the pipe from some convenient origin, the velocity gradient need no longer be written as the partial derivative of u with respect to y but is now equal to du/dy so that the shearing stress at any point

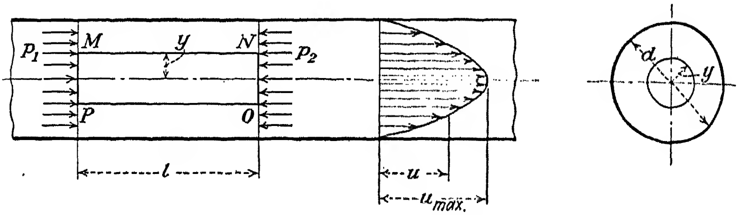


FIG. 128.—Laminar flow in a circular pipe.

is $\tau = \mu(du/dy)$. The total shear force on the outside of the cylinder is the product of the shear stress and the area of this cylindrical surface or

$$-\mu\frac{du}{dy}2\pi yl$$

The negative sign is necessary because, with the coordinate system here employed and with the velocity greatest at the center, du is a decrement.

Since the cylindrical body is in uniform motion, which is a condition of equilibrium, the sum of the axial forces is zero, or

$$(p_1 - p_2)\pi y^2 = -\mu\frac{du}{dy}2\pi yl$$

which may be simplified to give the differential equation

$$\frac{du}{dy} = -\frac{(p_1 - p_2)y}{2\mu l} \qquad (12)$$

This expression when integrated gives the result

$$u = -\frac{(p_1 - p_2)y^2}{4\mu l} + A$$

The constant of integration A is determined by the fact that there is no slip of fluid at the boundary so that, when $y = d/2$, $u = 0$, which requires that A have the value

$$A = \frac{(p_1 - p_2)\dfrac{d^2}{4}}{4\mu l}$$

The velocity at any point is thus found to be

$$u \quad \frac{(p_1 - p_2)}{4\mu l}\left(\frac{d^2}{4} - y^2\right) \tag{13}$$

indicating that the maximum velocity occurs at the center of the pipe and has the value

$$u_{\text{max.}} = \frac{(p_1 - p_2)d^2}{16\mu l} \tag{14}$$

Equation (13) also indicates that the distribution of velocity is in the form of a paraboloid of revolution or that for the longitudinal section shown in Fig. 128 it is a parabola. The average velocity may be readily determined by integration or more simply by recalling that the average height of a paraboloid such as that represented by Eq. (13) is one-half of the maximum ordinate. Thus the value of the average velocity is

$$V = \frac{(p_1 - p_2)d^2}{32\mu l} \tag{15}$$

The discharge or quantity of fluid passing any section in unit time is the product of the average velocity and the area, that is,

$$Q = \frac{\pi(p_1 - p_2)d^4}{128\mu l} \tag{16}$$

This equation is a mathematical statement of the Hagen-Poiseuille law. It indicates that the discharge from a pipe under steady laminar flow is directly proportional to the pressure difference on the ends of the pipe and to the fourth power of its diameter and is

inversely proportional to the viscosity and to the length of the pipe.

The validity of the Hagen-Poiseuille law has been well established not only by the experiments of the men whose names it bears but also by many other investigators. Because of the excellent agreement between theory and experiments, the latter serve as verifications of the hypotheses that the shearing stress in a viscous fluid is directly proportional to the velocity gradient and that the fluid in contact with a solid boundary must be at rest with respect to it.

It follows from the above discussion that laminar flow in a pipe is merely a continuous deformation of the fluid. In Fig. 129,

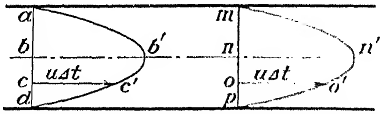


FIG. 129.—Result of laminar flow in a pipe.

for example, particles of fluid at a, d, m and p do not move, but such particles as b, c, n and o take the positions b', c', n' and o' after time t, so that the fluid which was in the cylinder $ampd$ is then bounded by the pipe and the paraboloids $ab'd$ and $mn'p$. In the same time any similar body of fluid in the pipe undergoes a like deformation.

86. Loss in Head in Laminar Flow through a Pipe.—The application of the Hagen-Poiseuille law to engineering work is usually concerned with the calculation of the loss in pressure or head due to viscosity. For a given discharge, Eq. (16) may be solved for $p_1 - p_2$, the result being

$$p_1 - p_2 = \frac{128Q\mu l}{\pi d^4} \tag{17}$$

This pressure difference may be expressed as a difference in head by dividing both sides of Eq. (17) by w, the specific weight of the fluid. Thus

$$h_1 - h_2 = \frac{128Q\mu l}{\pi d^4 w} \tag{18}$$

This last formula may be still further modified by substituting

ρg for w and then replacing the quotient μ/ρ by the kinematic viscosity ν, so that finally

$$h_1 - h_2 = \frac{128Q\nu l}{\pi d^4 g} \tag{19}$$

Problem 209. An oil having an absolute viscosity of 0.001 slugs per ft. sec. flows through a 1-in.-diameter pipe at an average velocity of 1.00 ft. per sec. Compute the pressure drop in 100 ft. of pipe and the velocity at a distance of 0.25 in. from the wall of the pipe.

210. A liquid having a viscosity of 250 centipoises flows through a pipe 10 cm. in diameter with an average velocity of 50 cm. per sec. Plot curves showing the distribution of velocity and shearing stress in a cross section of the pipe.

87. Motion of Bodies through a Nonturbulent Fluid. Stokes' Law.—The problems so far discussed in connection with the laminar motion of viscous fluids have, for the most part, been concerned with the movement of the fluid past stationary boundaries. Another group of problems which is of great importance treats of the resistance to motion produced by moving a solid body through a mass of fluid that was initially at rest. Such problems are frequently found in aeronautics and in marine and automotive engineering. The mathematical theory involved in the solution of these problems is extremely complicated and results have been obtained for only the simplest forms, such as the sphere and an infinitely long circular cylinder moving at right angles to its axis. No attempt will be made to discuss the theoretical details except to mention that the fundamental hypothesis on which the problems have been solved is that the motion is such that the inertia forces on the fluid particles may be neglected in comparison with the shearing forces due to viscosity.

In the case of a sphere moving with a constant velocity V through an infinite mass of fluid, it has been found that the resistance opposing the motion has the value

$$D = 3\pi\mu V d \tag{20}$$

where μ, as before, is the coefficient of viscosity and d is the diameter of the sphere. This formula is known as Stokes' law and was first demonstrated by him in 1851. Experimental data on the resistance of spheres will be discussed in detail in Chap. XII, but at this point it may be mentioned that Stokes' law holds only for a very restricted range of conditions. In the

case of ordinary fluids, such as water and air, the size of the sphere must be so small as to be practically microscopic in character, while with larger spheres, either the fluid must be extremely viscous or the velocity must be very low. These latter cases are often referred to as "creeping" motions.

In spite of these restrictions Stokes' law has not been without its practical applications. For example, it forms the basis for one method of measuring viscosity and has also been used to advantage in investigating the settling out of material suspended in liquids and in solving problems in diffusion. A recent application of much interest is concerned with the amount of ash deposit from stack gases in the vicinity of coal-burning power plants.[1]

88. Numerical Values of Viscosity.—In using the equations of the previous articles in this chapter, it is necessary to have some numerical data on viscosity. Air, water and castor oil are examples of fluids of low, medium and high viscosity. The accepted values of the absolute and kinematic viscosities of these fluids under standard conditions are given in Table IV.

TABLE IV.—VISCOSITY OF AIR, WATER, AND CASTOR OIL AT STANDARD CONDITIONS OF 59°F., 29.92 IN. HG (15°C., 760 MM. HG)

Fluid	Absolute viscosity, μ		Kinematic viscosity, ν	
	Poises	Slugs per ft. sec.	Stokes	Sq. ft. per sec.
Air.......	1.783×10^{-4}	3.723×10	1.455×10	1.566×10^{-4}
Water....	1.144×10	2.389×10^{-6}	1.145×10	1.232×10^{-5}
Castor oil	15.14	3.16×10	15.70	1.690×10^{-2}

The values of μ in poises given in Tables IV, V and VI are taken from "International Critical Tables," McGraw-Hill Book Company, Inc., New York, 1926–1930.

The range of values of absolute viscosity for a few common liquids at room temperature is indicated below in poises.

Liquid	Viscosity, Poises
Gasoline..............................	0.003–0.006
Kerosene..............................	0.02
Light lubricating oils..................	0.025–1.5
Medium lubricating oils................	1.5–3.5
Heavy lubricating oils..................	3.5–20

[1] CROFT, H. O., The Calculation of the Dispersion of Flue Dust and Cinders from Chimneys, *Trans. A.S.M.E.*, vol. 57, pp. 5–10, 1935.

Viscosities of a few common gases in poises are as follows:

Gas and temperature	Viscosity, poises	Gas and temperature	Viscosity, poises
Hydrogen at 20.8°C.	0.000089	Steam at 100°C.	0.000127
Helium at 21.4°C.	0.000199	Steam at 207°C.	0.000168
Nitrogen at 23°C.	0.000177	Methane at 100°F.	0.000115
Oxygen at 23°C.	0.000204	Carbon dioxide at 21°C. . .	0.000148

89. Effects of Temperature on Viscosity.—The effects of changes in temperature and pressure on the viscosity of fluids are best studied separately. It has already been mentioned in Chap. I, Art. 10, that liquids and gases are oppositely affected by changes in temperature, liquids exhibiting a decrease and gases an increase in viscosity with an increase in temperature. Water and air may be taken as examples of a liquid and a gas. The fact that they behave in opposite ways under a change in temperature is verified by the data given in Table V. Empirical

Table V.—Viscosity of Air and Water at Various Temperatures (Pressure = 760 Mm. Hg)

Temperature, °C.	Water		Air
	$\mu \times 10^2$ poises	Density, g. per cc.	$\mu \times 10^4$ poises
0	1.793	0.9998	1.709
20	1.008	0.9982	1.808
40	0.653	0.9922	1.904
60	0.469	0.9832	1.997
80	0.357	0.9718	2.088
100	0.283	0.9584	2.175
200	. . . . .	. . . . . .	2.582
300	. . . . .	. . . . . .	2.946
400	. . . . .	. . . . . .	3.277
500	. . . . .	. . . . . .	3.583

formulas representing the variation of the viscosity of water and air are also available.[1] Helmholtz, using Poiseuille's experiments, found that for water

[1] Lamb, H., "Hydrodynamics," 5th ed., p. 545, Cambridge University Press, 1924.

$$\mu = \frac{0.01779}{1 + 0.03368t + 0.000220099t^2} \tag{21}$$

while for air, according to Grindley and Gibson, the viscosity is

$$\mu = 0.0001702(1 + 0.00329t + 0.0000070t^2) \tag{22}$$

In these expressions t is the temperature in degrees centigrade and μ is the viscosity in poises.

The variation of viscosity with temperature is especially important in the case of lubricating oils since an oil that is satisfactory at the beginning of operation of a bearing may be insufficiently viscous when the full operating temperature is reached. On the other hand, an oil that is suitable for high-temperature work may have such a high viscosity at low temperatures that considerable difficulty may be experienced in starting the machine after it has been allowed to cool off. The effect of temperature on oil is illustrated by the figures on castor oil given in Table VI.

TABLE VI.—VISCOSITY OF CASTOR OIL AT VARIOUS TEMPERATURES (PRESSURE = 760 MM. HG)

Temperature, °C.	μ, poises	Density, g. per cc.
5	37.60	0.9707
10	24.18	0.9672
15	15.14	0.9638
20	9.86	0.9603
25	6.51	0.9569
30	4.51	0.9534
35	3.16	0.9499
40	2.31	0.9465
65.6	0.605	0.9284
100	0.169	0.9050

90. Effect of Pressure on Viscosity.—The effect of changes of pressure on the viscosity of fluids is practically negligible under ordinary conditions. Some liquids, such as ether and benzene, show a very slight increase while water exhibits a decrease in viscosity with increasing pressure, but in all these cases the changes are unnoticeable except at extremely high pressures. Liquid carbon dioxide behaves in much the same manner as ether

and benzene, but the changes in viscosity are appreciable in magnitude only in the neighborhood of the critical temperature where the liquid becomes gaseous in form. According to the kinetic theory of gases, the viscosity of such fluids is independent of the pressure and this statement is corroborated by the available experimental evidence, with the exception that the viscosity shows rather large variations in value when the gas is near its critical temperature. In the case of lubricating oils, the viscosity increases with the pressure much faster for mineral oils than for the fixed oils, a fact which is of considerable importance in the operation of bearings under heavy loads. The effect of pressure on the viscosity of Mobiloil "A" is shown by the following data.[1]

Gage pressure, kg. per sq. cm.	μ, poises
0	0.495
74.25	0.578
227.6	0.755
550.5	1.519
864.6	3.355
1019	5.105
1134	7.630
1164	10.950

91. Dimensional Homogeneity.—The general problem of fluid mechanics is to determine and to describe completely the nature of the flow of a fluid with respect to a certain specified arrangement of boundaries. As a part of this problem, it is frequently desirable to obtain an expression relating the fundamental quantities that characterize the fluid and its motion. Thus, earlier in this chapter formulas were given for the loss in head involved in the laminar flow of a viscous fluid through a circular pipe and for the resistance experienced by a sphere when moving with a constant velocity through a viscous fluid. In the majority of flows, particularly those concerned with viscous fluids, it is not possible to derive such relationships by analytical methods and it then becomes necessary to resort to experimental means. If, however, some indication is given of the important

[1] LANDOLT-BÖRNSTEIN, "Physikalisch-chemische Tabellen," vol. I, p. 169, Julius Springer, Berlin, 1923.

quantities on which the flow depends and whether or not there is any particular combination of them which is significant, the experimental work can be more readily systematized, and in some cases considerably reduced in amount.

The process of determining the proper combination of the significant physical quantities in any flow may be carried out by the methods of dimensional analysis. The basic principle of this method is that any equation relating physical magnitudes must be dimensionally homogeneous. In other words, if a formula is to represent an expression for resistance to the motion of a body through a fluid, then, since the resistance is a force, the quantities on which this force depends must be so arranged that their combination will also have the dimensions of a force. In applying this method it is usually convenient to reduce all quantities to the fundamental units of mass, length and time. These units are designated as fundamental because no one of them involves either of the others. If the problem at hand involves three distinct quantities, then, since there are three fundamental dimensions, it is possible to determine completely the form of their combination. If there are four quantities, three of them can be expressed in terms of the fourth but the problem cannot be solved completely except by more detailed analytical or experimental methods.

92. Application of Dimensional Analysis to Pipe Flow.—As an example of an application of the methods of dimensional analysis, the problem of the head lost due to friction in a circular pipe, as studied in Arts. 85 and 86, will be considered. Equation (18) of Art. 86 for the loss in head was

$$h_1 - h_2 = \frac{128Q\mu l}{\pi d^4 w} \tag{23}$$

Since the discharge is $Q = \pi d^2 V/4$, and $w = g\rho$, this may also be written in the form

$$h_1 - h_2 = \frac{32V\mu l}{d^2 \rho g} \tag{24}$$

The form of this equation can be found by dimensional analysis if it is assumed that the loss in head is dependent on the mass density, the viscosity, the velocity and the pipe diameter. Equation (24) also contains the length of the pipe but this may be

expressed as a constant times the diameter. The acceleration g which appears in Eq. (24) will be omitted because of the fact that neither inertia nor gravitational forces are involved in the flow as it was originally set up.

It is now assumed that the loss in head along the pipe may be written in the form

$$h_1 - h_2 = f(\rho, \mu, V, d)$$

or

$$h_1 - h_2 = k\rho^a\mu^b V^c d^e \tag{25}$$

in which k is a nondimensional coefficient of proportionality and a, b, c and e are undetermined exponents. The left side of Eq. (25) represents a head which is measured in units of length. Expressing all the quantities in this equation in terms of the fundamental units M, L and T, the following dimensional equality is obtained:

$$L = \left(\frac{M}{L^3}\right)^a \left(\frac{M}{LT}\right)^b \left(\frac{L}{T}\right)^c L^e \tag{26}$$

Inasmuch as the fundamental dimensions M, L and T are independent, Eq. (26) can be satisfied only by equating the exponents of the corresponding terms. In this way three separate equations are obtained, which are

$$\begin{aligned} 1 &= -3a - b + c + e \\ 0 &= a + b \\ 0 &= -b - c \end{aligned}$$

These expressions may be solved simultaneously for three of these exponents in terms of the fourth. If all are expressed in terms of b, then

$$\begin{aligned} a &= -b \\ c &= -b \\ e &= 1 - b \end{aligned}$$

so that Eq. (25) for the loss in head now becomes

$$h_1 - h_2 = kd\left(\frac{\mu}{\rho V d}\right)^b \tag{27}$$

Since k is nondimensional and since the right side of this expression should have the dimension of a length, it is obvious that the

combination of terms $\mu/\rho V d$ should be nondimensional. That this is the case may be verified by direct substitution of the dimensions of the quantities involved. But Eq. (27) does not appear to correspond in form to the original Eq. (24). However, the nondimensional combination $\mu/\rho V d$ may be introduced into the latter, giving as a result

$$h_1 - h_2 = 64\frac{V^2}{2g}\left(\frac{l}{d}\right)\left(\frac{\mu}{\rho V d}\right) \tag{28}$$

In this form the velocity head $V^2/2g$ has the linear dimension L, while the last two factors are dimensionless, so that the equation is still dimensionally homogeneous.

93. The Reynolds' Number.—The application of the methods of dimensional analysis to the pipe-flow problem does not give the result in the same form as that obtained by direct analysis. However, it does bring out the fact that there is a nondimensional combination of the physical quantities which describe the flow and on which the loss in head is dependent. This combination is known as the Reynolds' number after Osborne Reynolds, who was the first to show its meaning and importance. It will hereafter be designed by the symbol

$$N_R = \frac{\rho V d}{\mu} \tag{29}$$

Equation (28) thus indicates that, for a given value of the Reynolds' number, the loss in head in a pipe is directly proportional to the velocity head and to the ratio of length to diameter. If dimensional analysis had not suggested that the quantity N_R was of some significance, this simple form for the loss in head might have remained hidden in the form in which Eq. (24) is expressed. When both the Reynolds' number and the length-diameter ratio are constant, the loss in head is proportional only to the velocity head. With l/d constant, such flows have geometrically similar boundaries while, as will be shown in Chap. XV, a constant value of N_R indicates so-called "dynamic similarity" of the flows, that is, the forces acting on the fluid particles are in the same ratio at corresponding points.

In studying the motion of bodies through a fluid, it is found that the resistance is dependent on a combination of quantities which is analogous to the Reynolds' number for flow in pipes. The Reynolds' number for pipe flow is proportional to the diam-

eter; in the case of the moving body this term is replaced by some convenient length which is indicative of the size of the body. In many cases this quantity is taken as the length of the body measured in the direction of its motion. In all types of viscous fluid flows, the motion is found to be dependent on a general form of the Reynolds' number which may be written as

$$N_R = \rho Vl \tag{30}$$

in which l represents some characteristic dimension of the body or bounding surfaces.

The significance of Reynolds' number in the case of a body moving through a viscous fluid may be exemplified by means of Stokes' law for the resistance of a sphere. If the value of the resistance given by Eq. (20) is multiplied and divided by the density, velocity and diameter, the result is

$$D = \frac{3\pi\rho\mu V^2 d^2}{\rho V d} = 3\pi\rho V^2 d^2\left(\frac{\mu}{\rho V d}\right) = \frac{3\pi\rho V^2 d^2}{N_R} \tag{31}$$

In this case the Reynolds' number is $N_R = \rho V d/\mu$, the diameter of the sphere having been taken as its characteristic length. The impression should not be obtained from Eq. (31) that the drag of the sphere is proportional to the square of the velocity. This difficulty will be avoided if it is remembered that the Reynolds' number contains the velocity to the first power.

94. The Critical Reynolds' Number.—In the earlier discussion of Reynolds' experiments on the flow of water through circular pipes, it was mentioned that he found that under certain conditions the flow would be laminar in character and that under others it would be turbulent. It is now possible to discuss quantitatively the criterion which determines the type of flow that takes place under any given set of circumstances. The results of Reynolds' original experiments, as well as those of many other workers in this field, particularly those of Schiller,[1] have shown that at a certain so-called critical value of the Reynolds' number, based on the pipe diameter and the average velocity, the flow begins to exhibit turbulence. Thus for a pipe of a given diameter and carrying a fluid of a certain viscosity, the flow is laminar until the velocity reaches the value correspond-

[1] SCHILLER, L., *Forschungsarbeiten Ver. deut. Ing.*, vol. 248, p. 16.

ing to the critical Reynolds' number. For slightly higher speeds the flow becomes turbulent at a point at a considerable distance down the pipe from the inlet, but this turbulence may be of an intermittent character and does not extend throughout the entire remaining length of the pipe. As the velocity is still further increased, the turbulence becomes more and more complete until finally the entire length of the pipe beyond the point where turbulence first commenced is filled with fluid in which the motion is of this type. It appears that the change from laminar to turbulent flow is not an instantaneous one but occurs more or less gradually over a range of values of Reynolds' numbers known as the transition range. The Reynolds' number at which laminar flow ceases to exist is known as the critical Reynolds' number or Reynolds' criterion. It will be designated by N_C.

Although considerable information is now available concerning the effects of turbulent flow on the velocity distribution, pressure drop and other external characteristics of the motion, very little is known about the actual mechanism by means of which turbulence is produced. The most satisfactory explanation offered is that in any flow there are initially certain small disturbances and that when laminar flow exists, these disturbances are rapidly damped out, while turbulent flow is the result of their augmentation. While this is a very sketchy and incomplete attempt to explain the fundamental nature of the difference between these two kinds of flows, it does give some insight into their processes of formation. From this point of view it would appear that conditions in the fluid before and at the inlet to the pipe would have considerable influence on the motion, and this supposition is borne out by the evidence obtained from experiments. If the fluid in the reservoir which feeds into the pipe is allowed to stand for several days before any tests are made, so that it has attained an almost complete state of rest, it is possible to obtain much higher values for the critical Reynolds' number than would otherwise be the case. Likewise a carefully flared and polished mouthpiece placed on the inlet of the pipe makes it possible to increase the value of the critical Reynolds' number. The most reliable experiments indicate that there is no definite upper limit to this critical value and values have been obtained ranging all the way from about 2400 to 50,000, depending on the care taken

to minimize the effects of initial disturbances. However, it does appear that the critical Reynolds' number has a lower limit; that is, there is a value below which turbulent flow does not persist and below which the motion is always laminar in character, even for extremely violent initial disturbances. The value of this "lower critical Reynolds' number," as determined by Schiller, is 2320.

When the Reynolds' number for the flow in a circular pipe exceeds the lower critical value of 2320, there is always the possibility that the flow may be turbulent rather than laminar, so that for values higher than 2320 the Hagen-Poiseuille law should not be applied unless there is evidence that the critical value for the installation in question is higher than this lower limit. Since the great majority of actual flows found in engineering work have proved to be turbulent in character, turbulent flow is sometimes known as hydraulic flow. Its detailed study in the case of pipes will be taken up in the next chapter.

Turbulence and the transition from laminar to turbulent flow are of importance not only in connection with the flow in pipes but in many other examples of fluid motion. An outstanding instance is found in the so-called boundary-layer theory for the determination of the resistance of bodies moving through fluids of small viscosity such as air or water. In this theory the effects of viscosity are confined to a thin film of fluid adjacent to the surface of the body and in this layer the flow may be of either a laminar or a turbulent character. The nature of the flow has, as might be expected, a marked effect on the resistance of the body. A fuller discussion of such problems will be found in Chap. XII.

Example. An oil having a specific gravity of 0.78 and an absolute viscosity of 0.075 poises flows through a horizontal $\frac{1}{2}$-in.-diameter pipe line 40 ft. long. Determine the highest average velocity for which the flow is certain to be laminar and compute the pressure difference necessary to maintain this flow.

Solution.—The absolute viscosity of the oil is

$$\mu = 0.075 \text{ poises} = 0.075 \frac{\text{g.}}{\text{cm.sec.}}$$

and, since 1 g. = 6.85×10^{-5} slugs (see page 170) and

$$1 \text{ cm.} = \frac{1}{2.54 \times 12} = 0.0328 \text{ ft.,}$$

then

$$\mu = \frac{0.075 \times 6.85 \times 10^{-5}}{0.0328} = 1.566 \times 10^{-4} \frac{\text{slugs}}{\text{ft.sec.}}$$

The limiting velocity for laminar flow corresponds to the condition where the Reynolds' number is equal to its critical value, that is,

$$\frac{\rho V d}{\mu} = N_C$$

or

$$V = \frac{\mu N_C}{\rho d}$$

Now $N_C = 2320$ and

$$\rho = \frac{w}{g} = \frac{0.78 \times 62.4}{32.2} = 1.51 \text{ slugs/cu. ft.}$$

Then

$$V = \frac{2320 \times 1.566 \times 10^{-4}}{1.51 \times \frac{0.5}{12}} = 5.77 \text{ ft./sec.}$$

The pressure drop in terms of the average velocity, as obtained from Eq. (15), is

$$p_1 - p_2 = \frac{32 V \mu l}{d^2} = \frac{32 \times 5.77 \times 1.566 \times 10^{-4} \times 40}{\left(\frac{0.5}{12}\right)^2}$$

$$= 667 \text{ lb./sq. ft. or } 4.63 \text{ lb./sq. in.}$$

Problem 211. Water at 40°C. flows through a 1-in.-diameter pipe. What is the lowest velocity at which the flow can be turbulent? What is the corresponding value for air at atmospheric pressure and 40°C.?

212. Compute the highest velocity at which the flow of water in a ½-in. pipe is certain to be laminar at zero and 100°C.

213. Compute the Reynolds' number for a sphere 30 in. in diameter in atmosphere at 15°C. moving at a velocity of 60 m.p.h.

95. Surfaces of Discontinuity and Vortex Formation. The forces due to viscosity are present in real fluids only when there exists a velocity gradient in the direction normal to the motion of the fluid. This is immediately evident from Newton's law for the shearing stress between adjacent layers of fluid as written in the form of Eq. (2). The layers of fluid dealt with there are of infinitesimal thickness and there is a continuous variation in velocity from layer to layer. In many cases circumstances may arise which lead to the presence of two adjacent layers of fluid having velocities differing from each other by a finite amount. The velocity distribution and its gradient normal to the motion

can no longer be regarded as continuous functions and the theory of viscous fluids developed in the preceding pages is not strictly applicable. There is a marked difference in the magnitudes of the velocity on the two sides of the surface separating the two layers of fluid, and for this reason this surface is commonly known as a surface of discontinuity.

In general, surfaces of discontinuity are very unstable and remain intact for only a short period of time during the early stages in the development of a flow. The surface rolls up into a series of eddies or vortices which adjust themselves so as to form a stable arrangement. The development of these eddies from a surface of discontinuity has been explained by Prandtl[1] and is best illustrated by means of an example.

(a)-Symmetrical Flow without Discontinuity

(b)-Asymmetrical Flow with Discontinuity

FIG. 130. Forming of surface of discontinuity.

The streamlines of the two-dimensional flow of a uniform stream past a strut having a symmetrical cross section are shown in Fig. 130*a*. In this case the direction of the undisturbed flow is parallel to the axis of symmetry of the section and the fluid layers passing along the upper and lower surfaces meet at the trailing edge with equal velocities so that no surface of discontinuity is formed. If the direction of the flow is inclined upward, then the layer of fluid passing along the upper surface reaches the trailing edge with a lower velocity than that of the layer passing along the lower surface as shown in Fig. 130*b*. A surface of discontinuity is thus formed in the fluid between these two layers after they leave the trailing edge of the section.

Initially the surface of discontinuity leaves the trailing edge of the strut as a smooth line, but through the action of any small disturbances it soon acquires a wavy formation. The breaking down of this flow into a series of eddies may be shown by using a system of axes which moves with a velocity equal to the mean

[1] EWALD, P. P., PÖSCHL, T., and PRANDTL, L., "The Physics of Solids and Fluids," pp. 225–227, Blackie & Son, Ltd., London, 1930.

velocity of the two layers of fluid. The wave system advances with this same velocity relative to the strut so that, with respect to the new coordinate system, the waves are at rest as shown in Fig. 131*a*. The heavy line represents the wavy surface of discontinuity and the fluid below it is moving to the left while that above it moves to the right. From the character of the streamlines, it is possible to locate at once the regions of increased and decreased pressure, these being indicated by the plus and minus signs in the figure. Because of these differences in pressure, the waviness of the surface of discontinuity increases, taking on the shape shown in Fig. 131*b* and finally rolling up into eddies as shown in Fig. 131*c*.

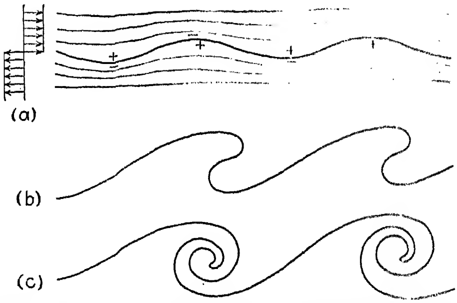


FIG. 131.—Surface of discontinuity breaking down into eddies.

In all actual flows where discontinuities may appear, the fully developed flow consists of a system of eddies which has adjusted itself in strength and position so as to form a stable arrangement. The wake behind a normal plate or bluff-shaped body is represented by a double row of eddies known as a vortex trail. Cases where an isolated eddy or vortex is formed in a fluid are frequently found in nature, the tornado and waterspout being two of the best known examples.

96. Properties of Vortices.—An eddy or vortex formed in a real fluid consists of a relatively small core rotating about an axis within itself as though it were a solid rod. The action of viscosity on its outer surface transmits this rotation to the surrounding fluid and causes the latter to circulate around the core with a velocity that varies more or less inversely with the distance from its center. In hydrodynamic theory where viscosity is neglected, there is also a flow known as a vortex, which may be

defined by saying that the velocity is inversely proportional to the distance r from its center, that is,

$$V = \frac{\Gamma}{2\pi r} \tag{32}$$

The constant Γ is known as the strength of the vortex and is equal to the circulation around it. The fluid moves around the center on concentric circular streamlines and there is no core as in the actual vortex. The velocity at the center of the mathematical vortex consequently becomes infinitely large, a condition which could not exist in nature, while the velocity approaches zero only at an infinitely great distance from the center. In the

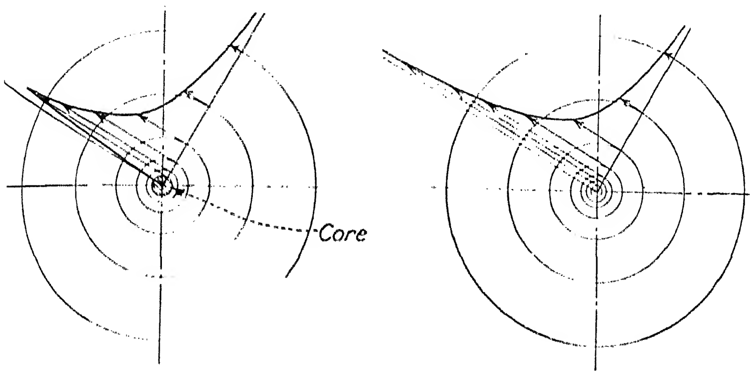


FIG. 132. Flow due to a real and a mathematical vortex.

real vortex the velocity drops off somewhat more rapidly than would be indicated by Eq. (32) because of the effect of viscosity. The nature of the flow in a right cross section of a real and mathematical vortex is shown in some detail in Fig. 132. Equation (32) may usually be employed for approximate calculations of the velocity field around a vortex which has an infinitely long core in the form of a straight line, provided that such information is not wanted for points in the immediate neighborhood of the core. When the core is curved the calculations are more complicated and will not be discussed here. It might be mentioned, however, that the problem is mathematically identical with one in electricity, that is, the determination of the strength of the magnetic field around an infinitesimal conductor carrying a constant current. The velocity outside the vortex core corre-

sponds to the strength of the magnetic field and the strength of the vortex to the magnitude of the current passing through the conductor.

Helmholtz, who was the originator of the mathematical theory of vortex motion, demonstrated several laws concerning their behavior. These laws are as follows:

1. A vortex is of constant strength throughout its entire length and cannot terminate at a point within the fluid, except at a boundary.
2. The strength of a vortex remains constant with time.
3. A vortex always consists of the same fluid particles as it moves through the fluid.

The first of these laws would imply that, in the case of the tornado, the vortex must extend upward an infinite distance from the surface of the earth. However, in a real fluid, such as air, this is not necessarily true because the action of viscosity will produce a dissipation of energy in the vortex and cause it to disintegrate at a finite distance from its boundary. For calculations of the velocity near the earth's surface produced by the tornado, the assumption that the vortex obeys Helmholtz' first law will give reasonably accurate results. The second and third laws are also not strictly true for real vortices, and again the differences may be attributed to the effects of viscosity. An example of a vortex whose core is in the form of a closed curve is the smoke ring, which is easily produced for demonstration purposes.

Many fruitful applications of the theory of vortices have been made in problems of fluid mechanics, particularly in the calculation of the resistance of bluff bodies, using the idea of the vortex trail mentioned above, and in the aerodynamics of airplane wings.

The circulatory flow used in the study of the rotating cylinder in Chap. VII may be considered as being obtained by the superposition of a uniform flow on that produced by a vortex located at the center of the cylinder cross section. This vortex does not exist in the actual flow but may be regarded as producing an effect which is theoretically the same as that produced by the cylinder.

Further light may also be thrown on the concept of the surface of discontinuity by regarding it as being made up of a number of infinitesimally small vortex filaments all rotating in the same

direction. When a uniform stream of fluid passes over such a vortex layer, it is at once evident from Fig. 133 that a discontinuity of velocity will result. The vortices appear to act as a series of roller bearings between the two layers. When infinitesimal vortices are distributed continuously throughout a body of fluid, the latter is said to possess vorticity. Such a concept is sometimes used for studying the behavior of the turbulent

FIG. 133. Surface of discontinuity represented by a vortex layer.

motion of viscous fluids. Several of these problems will be treated in greater detail in later chapters of this book.

General Problems

214. A sphere is in a stream of gas having a kinematic viscosity of 10 stokes and a velocity of 20 ft. per sec. Its Reynolds' number is 200. What is the Reynolds' number of the same sphere in a stream of air at 20°C. and a velocity of 40 m.p.h.?

215. Compute the velocity at which castor oil at 15°C. will flow through a tube 0.4 cm. in diameter and 4 ft. long under a head of 18 in. What is the Reynolds' number? What head would be required to reach the critical velocity?

216. A vertical shaft 2 in. in diameter rotates at 240 r.p.m. in a concentric cylindrical bearing 2.5 in. in diameter. The bearing contains lubricating oil having a viscosity of 0.495 poises. Assuming a straight-line distribution of velocity in the oil, what is the shear stress in the oil next to the shaft? If 1 ft. of length of the shaft is in oil, what is the torque on the shaft due to the oil?

217. A capillary tube 0.2 cm. in diameter and 10 cm. long discharges 1.0 liter of liquid in 10 min. under a pressure difference of 2 in. of mercury. If the specific gravity of the liquid is 0.9, what is the viscosity?

CHAPTER IX

FLOW OF FLUIDS IN PIPES

97. Motion of Fluid in a Pipe.—A fluid in a pipe is constrained laterally by pressure forces at the walls so that the general motion of the fluid must be along the axis of the pipe. These lateral forces, being normal to the direction of motion, can have no effect on the velocity. The velocity of the flow is controlled and maintained by the axial forces, namely, the pressure difference and the axial component of gravity forces. An idea of the manner in which these forces act may be gained by considering a cylindrical body of fluid contained in a portion of pipe of length

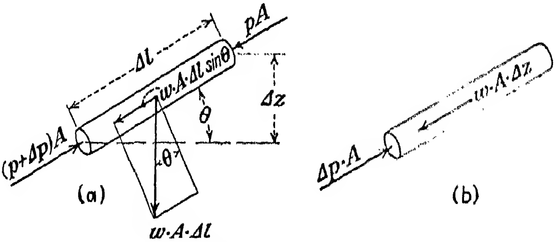


FIG. 134.—Body of fluid in a pipe.

Δl and cross section A, such as that shown in Fig. 134*a*. It is affected by an axial force due to the difference in pressure forces, $\Delta p\ A$, and the axial component of weight, $wA\ \Delta l \sin\theta = wA\ \Delta z$; the simplified force condition is as shown in Fig. 134*b*. When $\Delta p\ A = wA\ \Delta z$, the fluid is static, that is, $\Delta p = w\ \Delta z$, which is the usual condition for static fluids. If $\Delta p > w\ \Delta z$, the flow will be upward and toward the right at such a velocity that the resistance to flow is downward and equal to

$$R = \Delta p\ A - wA\ \Delta z = A(\Delta p - w\ \Delta z)$$

When $\Delta p < w\ \Delta z$, the flow is down and toward the left at such a velocity that the resistance is upward and equal to

$$R = wA\ \Delta z - \Delta p\ A \quad A(w\ \Delta z - \Delta p)$$

Figure 135 represents a portion of a pipe line having a length l and a constant cross-sectional area A. Points 1 and 2 are at elevations z_1 and z_2, respectively, above any horizontal datum plane BB. The Bernoulli constants for points 1 and 2 referred to this datum are

$$\frac{V_1^2}{2g} + \frac{p_1}{w} + z_1 = H_1 \tag{1}$$

and

$$\frac{V_2^2}{2g} + \frac{p_2}{w} + z_2 = H_2 \tag{2}$$

It will be recalled from Art. 43 that the terms of Bernoulli's constant when written in the above form represent energy

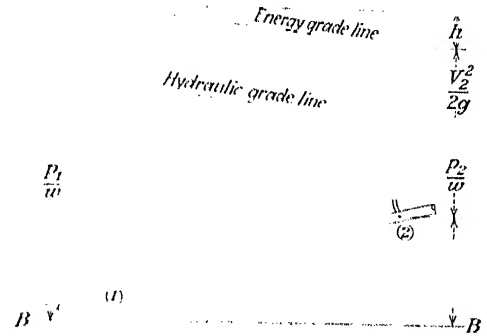


FIG. 135. Hydraulic grade line and energy grade line.

content of the several forms in foot-pounds per pound of fluid, a quantity having the dimension

$$\frac{\text{foot-pounds}}{\text{pounds}} = \text{feet}$$

The terms are therefore commonly called velocity head, pressure head and potential head, respectively, while their sum H is the total head. The difference in total heads at points 1 and 2 is then

$$H_1 - H_2 = \left(\frac{V_1^2}{2g} + \frac{p_1}{w} + z_1\right) - \left(\frac{V_2^2}{2g} + \frac{p_2}{w} + z_2\right) = h \tag{3}$$

If the fluid is a liquid or if the change in density is negligible, the velocity in a pipe of uniform diameter is constant and the

velocity heads in Eqs. (1), (2) and (3) are all equal. Then from Eq. (3)

$$H_1 - H_2 = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right) = h \tag{4}$$

The terms of Eqs. (3) and (4) appear in Fig. 135, and an examination of that figure shows that, in the case of a pipe carrying a liquid, h in this equation is the difference in elevation of the free surfaces in open piezometer tubes connected to the pipe at the points in question. A line connecting such free surfaces is known as the pressure grade line or hydraulic grade line, a very useful device in dealing with flow in pipes. The line drawn at a distance $V^2/2g$ above the hydraulic grade line is known as the energy grade line since its ordinate measured from the datum plane represents the total Bernoulli constant. The velocity being constant, the direction of flow in a pipe is always down the slope of the energy grade line and is independent of the slope of the pipe itself.

The fall in the energy grade line along a pipe is the loss of head or energy loss per pound of fluid. It is the energy which in the case of a liquid has been transformed to heat and no longer exists in any of the three forms represented by the terms of the Bernoulli constant. This loss, or, more properly, this transformation of energy, results from natural resistance to flow.

98. Nature of Resistance to Flow in Pipes.—The study of the nature of the resistance to flow in pipes and the gathering of data to be used in computing the loss of energy or head due to such resistance have occupied the time of great numbers of scientists and engineers.

The resistance to laminar flow of fluids is due entirely to viscosity. The loss of energy is commonly called the friction loss but it is not to be supposed that there is friction in the sense in which the term is used in dealing with relative motion of solids in contact. There is no motion or slip of the fluid at the walls of the pipe. As heretofore explained in Art. 85, laminar flow consists merely of a continuous change in shape of the body of fluid in the pipe at any time. The only resistance to this change in shape is due to viscosity of the fluid, which property must, therefore, be charged with all lost head or energy in laminar flow.

When the flow in a pipe is turbulent, the motion is so complex that no rigorous analysis or detailed description of the motion has been made. Particles of turbulent fluid may successively occupy different positions in the cross section of the pipe and are no longer confined to a particular lamina. There is then an interchange of momentum, not only between molecules of the fluid, but also between finite masses of fluid as they move to other parts of the cross section. It is certain that such finite masses do not long remain intact and that there is a very thorough mixing of the fluid. The resistance to turbulent flow is the combined effect of forces due to viscosity and those due to inertia.

An idea of the great difference between resistance to the two kinds of flow which may exist in pipes can be gained from the fact that in laminar flow the resistance or loss varies directly with the first power of velocity while in turbulent flow it varies approximately with the second power of velocity.

99. Reynolds' Number for Pipes. The significance of Reynolds' number as a criterion of the kind of flow in a pipe was discussed in Arts. 93 and 94. The expression $N_R = \rho Vd/\mu = Vd/\nu$ contains the factors which affect resistance to flow and for that reason has been referred to as a "least common denominator" of pipe flow.

The critical value of N_R, called Reynolds' criterion N_C, is the same for all fluids. All have laminar motion in a pipe when N_R is less than the critical value of 2320 and they may, and usually do, have turbulent motion when N_R is higher. This number is also a criterion for velocity distribution and the distance required for its adjustment in the case of laminar flow. Experiments on one fluid may be used to predict definitely the resistance to flow of any other fluid, even in a pipe of different diameter and at a different velocity, provided that N_R is the same for the two cases.

Two pipes of different diameters, containing fluids of different densities and viscosities moving at unequal velocities, may be likened to two similar triangles in that whatever they have in common must be dimensionless. In the triangles that thing is the angle or the ratio of corresponding sides; in pipes it is the dimensionless combination of factors known as Reynolds' number. The use of this quantity as a parameter in studying and plotting experimental data on the flow of fluids has done

much to advance the knowledge of the nature of such flow and also of the numerical constants involved because it serves to correlate the data on all fluids.

100. Velocity Distribution in Cross Section of a Pipe.—The nature of velocity distribution for laminar flow in pipes is susceptible to exact analysis. It was shown in Art. 85, in connection with the derivation of the Hagen-Poiseuille Law, that when the velocity is plotted against position on the diameter of a pipe the resulting curve is a parabola with its axis coinciding with the

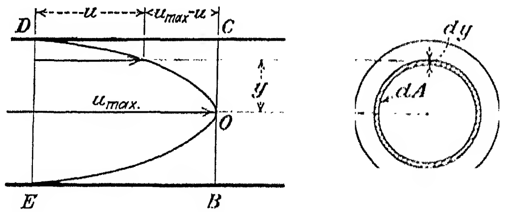


FIG. 136.—Ideal velocity distribution for laminar flow.

axis of the pipe, as in Fig. 136. The velocity at any distance from the axis of the pipe is given by Eq. (13), page 173, as

$$= \frac{(p_1 - p_2)}{4\mu l}\left(\frac{d^2}{4} - y^2\right) \tag{5}$$

The velocity at the center is maximum and, letting $y = 0$, it is found to be

$$u_{\text{max.}} = \frac{(p_1 - p_2)}{4\mu l}\frac{d^2}{4} \tag{6}$$

and from Eqs. (5) and (6)

$$\frac{4y^2}{d^2}\Big) u_{\text{max.}} \tag{7}$$

The flow across the ring-shaped element of area $dA = 2\pi y\,dy$ in Fig. 136 is

$$dQ = u\,dA = 2\pi u y\,dy$$

Substituting u as given in Eq. (7) and integrating,

$$Q = \int dQ = 2\pi u_{\text{max.}} \int_0^r \left(1 - \frac{y^2}{r^2}\right) y\,dy = \pi r^2 \frac{u_{\text{max.}}}{2} \tag{8}$$

Expressing Q in terms of the average velocity V and the area of the pipe πr^2 and equating this value to that given by Eq. (8),

$$Q = \pi r^2 V = \pi r^2 \frac{u_{\max}}{2}$$

whence the average velocity is

$$V = \frac{u_{\max}}{2} \tag{9}$$

This could have been deduced from the fact that the volume of the paraboloid DEO is half of the volume of the circumscribed cylinder $BCDE$.

The ideal parabolic distribution of velocity exists only at a considerable distance downstream from the entrance to the pipe and at values of N_R not too close to the critical value N_c. The

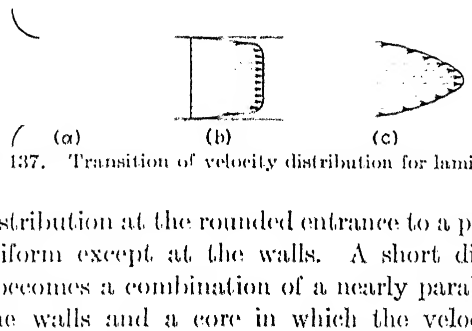


FIG. 137. Transition of velocity distribution for laminar flow.

velocity distribution at the rounded entrance to a pipe is approximately uniform except at the walls. A short distance downstream it becomes a combination of a nearly parabolic distribution at the walls and a core in which the velocity is nearly uniform. Thus there is a gradual transition from the distribution shown in Fig. 137*a* to the ideal parabolic distribution of Fig. 137*c*. The distance from the entrance required for practically complete transition has been shown theoretically by Boussinesq and experimentally by Nikuradse[1] to be such that $x/N_Rd = 0.065$, in which x is the distance from the entrance and d is the diameter. Here again N_R plays an important part since the dimensionless expression x/N_Rd is a criterion for velocity distribution; that is to say, the variation of velocity in a cross section is the same in different pipes for equal values of x/N_Rd.

[1] PRANDTL, L., and O. J. TIETJENS, "Applied Hydro- and Aero-mechanics," p. 25, McGraw-Hill Book Company, Inc., New York, 1932. This work gives velocity distribution diagrams after Nikuradse and contains a thorough discussion of the subject.

With turbulent motion the distribution of velocity does not follow any simple mathematical curve. Many empirical formulas have been devised to express the relation between velocity and distance from the axis, each of which has some limitation.[1] Velocity distribution in turbulent flow is more nearly uniform than for laminar flow. At points well downstream from any disturbance and for values of N_R well above N_C, the average velocity in the cross section is

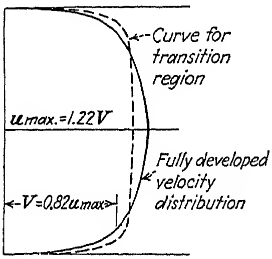


FIG. 138.—Velocity distribution of turbulent flow for N_R well above N_C.

$$V = 0.82u_{\text{max.}} \quad \text{or} \quad u_{\text{max.}} = 1.22V \quad (10)$$

The latter figure is sometimes found as high as 1.25 and 1.235 may be taken as an average value over a long range of N_R. The typical velocity distribution is established in a distance of 25 to 50 diameters from the entrance, this distance being shorter than for laminar flow and apparently independent of N_R. The velocity curves in the transition region do not differ so widely from that finally established as in the case of laminar flow. Figure 138 affords a comparison of the established velocity curve with one in the transition region and Fig. 139 shows a comparison of typical velocity curves for laminar and turbulent flow at the same average velocity.

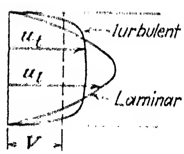


FIG. 139. Comparison of velocity distribution for turbulent and laminar flow at same average velocity.

Stanton,[2] in very carefully conducted experiments, found no evidence of any slipping of fluid along the wall of the pipe, and several others have confirmed his results. There is a very thin annular space at the wall in which the flow is laminar. The presence of this thin laminar sub-layer has been demonstrated experimentally and theoretically and Prandtl[3] shows that it has a thickness of $\delta = 125.4rN_R^{-7/8}$. Between this very thin laminar boundary layer and the center of the pipe the

[1] LEA, F. C., "Hydraulics," 5th ed., p. 203, Longmans, Green & Company, New York, 1930. This book gives a summary of empirical formulas for this purpose. See also Art. 107 of this text.

[2] STANTON, T. E., The Mechanical Viscosity of Fluids, *Proc. Roy. Soc.*, vol. 85, p. 366, 1911.

[3] PRANDTL and TIETJENS, *op. cit.*, p. 78.

flow is turbulent. It is not to be supposed that there is a sharp division between the laminar and turbulent portions of the cross section. On the contrary, turbulence diminishes gradually near the wall.

In either type of flow the velocity distribution is altered by any change in form or alignment of the pipe, by any convergence or divergence of the walls however gradual or by a temperature gradient across the pipe. The thorough mixing of fluid in turbulent flow makes a marked temperature gradient impossible except in the laminar sub-layer.

The effect of temperature gradient is more pronounced in laminar flow owing to the change of viscosity with temperature. If heat is being lost through the wall of the pipe, a liquid is more viscous at the wall than in the center and the ideal velocity diagram of Fig. 136 or 137*c* is distorted so as to be sharper at the center and to have a smaller slope at the wall. If heat is being absorbed by the liquid at the wall, the velocity diagram becomes more blunt at the vertex. The effect of temperature gradient on velocity distribution of a gas in laminar flow is directly opposite to its effect in the case of a liquid. Large temperature gradients across the pipe make accurate computation of resistance quite difficult if not impossible.

All changes in velocity distribution change the required energy content and may have considerable effect on the resistance to flow.

101. Relation of Kinetic Energy Content to Velocity Distribution. It has been demonstrated in previous articles that the kinetic energy per pound of fluid having velocity u is $u^2/2g$, the velocity head. The total kinetic energy contained in the fluid passing in 1 sec. through a cross section over which the velocity is uniformly V is the product of the total number of pounds per second and $V^2/2g$, or

$$\text{K.E. per sec.} = \frac{QwV^2}{2g} \text{ ft.lb./sec.}$$

When the distribution of velocity is nonuniform, the average velocity being V, the kinetic energy content will invariably be more than $QwV^2/2g$.

In laminar flow with parabolic velocity distribution, for example, it can be readily shown that the kinetic energy content

is $2QwV^2/2g$, or twice as much as for a uniform velocity of V. Referring to Fig. 136, the ring-shaped element of area is $\overline{dA} = 2\pi y\,\overline{dy}$ and the velocity through it is u, whence the elementary discharge is[1]

$$\overline{dQ} = u\,\overline{dA} = 2\pi uy\,\overline{dy} \tag{11}$$

The kinetic energy in the fluid passing $\overline{dA}$ each second is

$$w\,\overline{dQ}\frac{u^2}{2g} = \frac{\pi w}{g}u^3y\,\overline{dy} \tag{12}$$

From Eq. (7) and Eq. (9)

$$u = u_{\text{max.}}\left(1 - \frac{y^2}{r^2}\right) = 2V\left(1 - \frac{y^2}{r^2}\right)$$

Substituting this value of u in Eq. (12) and integrating over the entire area, that is, from $y = 0$ to $y = r$, the total kinetic energy per second is

$$\begin{aligned}\text{K.E. per sec.} &= \frac{8\pi wV^3}{g}\int_0^r\left(1 - \frac{y^2}{r^2}\right)^3 y\,\overline{dy}\\ &= \frac{\pi wV^3r^2}{g} = 2Qw\frac{V^2}{2g}\end{aligned} \tag{13}$$

From this it is seen that the total kinetic energy for parabolic distribution of velocity is twice as much as for a uniform velocity of V.

In the case of turbulent flow the total energy content is not conveniently found by integration at this time because the velocity distribution curve is not represented by a simple equation. For the purpose of arithmetic integration the cross section may be divided into small rings of area ΔA. The kinetic energy of the fluid passing through each ΔA in 1 sec. is then $\Delta A\ uw\frac{u^2}{2g} = \Delta E$ and the total kinetic energy content of the quantity per second is $\Sigma(\Delta E) = \Sigma\left(\Delta A\ uw\frac{u^2}{2g}\right)$. By taking values of u for each ΔA from the velocity distribution curve, it is found that the total kinetic energy per second for turbulent flow is about $1.1Qw\frac{V^2}{2g}$ or 10 per cent more than for uniform velocity.

[1] In terms of the type $\overline{dA}$ the symbol d denotes a differential, the bar being used to avoid confusion with diameter d.

This quantity is based upon the assumption that the local velocity is parallel to the axis of the pipe and that the motion does not vary with time. Neither of these assumptions is strictly true and the coefficient 1.1 is therefore somewhat small.

102. Energy or Head Lost in Pipes. Laminar Flow.—The head lost in laminar flow in a pipe can be computed directly from the Hagen-Poiseuille law, the only empirical data required being the viscosity and density of the fluid. Using this law in the form of Eq. (15), page 173,

$$V = \frac{(p_1 - p_2)d^2}{32\mu l} \tag{14}$$

in which d is the diameter and l is the length of pipe between points 1 and 2, the drop in pressure for a horizontal pipe is found to be

$$p_1 - p_2 = \frac{32\mu l V}{d^2} \tag{15}$$

and, dividing by the specific weight, the head lost is

$$h = \frac{p_1}{w} - \frac{p_2}{w} = \frac{32\mu l V}{wd^2} \tag{16}$$

Multiplying both numerator and denominator of this expression by ρVd and then substituting $N_R = \rho Vd/\mu$, the lost head in terms of Reynolds' number is seen to be

$$h = \frac{32}{wd^2} \cdot \frac{\mu}{\rho Vd} lV(\rho Vd) = \frac{32\rho lV^2}{wN_R d} \tag{17}$$

Noting that $\rho = w/g$ and canceling w,

$$h = \frac{64}{N_R} \frac{l}{d} \frac{V^2}{2g} \tag{18}$$

This form is considered very convenient because it contains the expression for velocity head $V^2/2g$ and also because it resembles the well-known Darcy formula. The pressure difference for horizontal pipes expressed in terms of velocity head and Reynolds' number is

$$p_1 - p_2 = \frac{64w}{N_R} \frac{l}{d} \frac{V^2}{2g} \tag{19}$$

While these formulas are convenient in the forms given in Eqs. (17), (18) and (19), they should not be allowed to obscure the fact that losses in laminar flow vary directly with the first power of V and not with V^2, since N_R contains V as a factor.

The above formulas for laminar flow, which are based on theoretical considerations, are well substantiated by experiment. The only empirical data required is the viscosity since resistance to laminar flow is independent of the roughness of the walls of the pipe if it is not so great as to cause an actual change in the interior dimensions or shape.

103. Energy or Head Lost in Pipes. Turbulent Flow.—The head lost in turbulent flow in pipes is computed from formulas involving the velocity, the length and diameter of the pipe, and the viscosity and density of the fluid. The formulas have a fairly rational basis but invariably depend for their usefulness upon an empirical coefficient. One of the most generally accepted of these is that by Darcy,[1]

$$h = f\frac{l}{d}\frac{V^2}{2g} \tag{20}$$

in which the terms l, d and V are as previously defined and f is a dimensionless coefficient. The latter varies widely with the condition and diameter of the pipe and the velocity. Many experimenters have collected data from which f has been computed and a few values for water under ordinary conditions are given in Table VII. Most of these experiments were performed on flow of water in pipes and, while their number is large, most of the experiments made with water are included in a comparatively short range of Reynolds' numbers.

It has been shown definitely that values of f for turbulent flow are closely related to Reynolds' number, but this relationship is such that it cannot be contained in one simple mathematical expression. Empirical formulas, notably those of Blasius and Lees, have been devised which express the relationship between f and N_R, but to cover the whole range of available experiments and engineering problems it is convenient to resort to a graph in which values of the coefficient f are plotted against values of N_R.

[1] Also variously credited to Chezy, Eytelwein, Weisbach and Fanning.

TABLE VII.—VALUES OF f FOR CAST-IRON PIPE CARRYING WATER

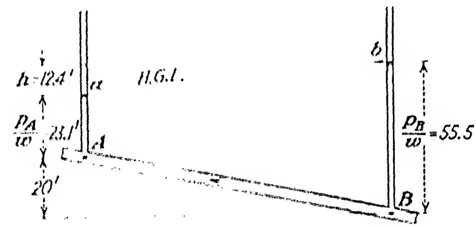
Velocity	Diameter, in.							
		6	8	12	16	18	20	24
1	0.028	0.027	0.026	0.025	0.024	0.023	0.022	0.021
2	0.027	0.026	0.025	0.024	0.022	0.022	0.021	0.020
4	0.025	0.024	0.023	0.022	0.021	0.020	0.020	0.019
6	0.024	0.023	0.023	0.021	0.020	0.020	0.019	0.018
8	0.023	0.022	0.022	0.021	0.020	0.019	0.019	0.018
10	0.023	0.022	0.021	0.020	0.019	0.019	0.018	0.018
12	0.022	0.022	0.021	0.020	0.019	0.019	0.018	0.018
16	0.022	0.021	0.021	0.020	0.019	0.018	0.018	0.017
Old pipe All velocities	0.049	0.046	0.044	0.042	0.040	0.038	0.037	0.036

Example.—A line of clean 6-in. cast-iron pipe shows a pressure of 10 lb. per sq. in. at A and 24 lb. per sq. in. at B, which is 600 ft. from A and 20 ft. lower. What is the quantity flowing and the direction of flow in the line?

Solution.—First compute $p_A/w = 10 \div 0.433 = 23.1$ ft. and $p_B/w = 55.5$ ft. Now, after sketching the hydraulic grade line, ab, it is evident that the flow is from B to A and that the head lost is $55.5 - 20 - 23.1 = 12.4$ ft. Then

$$h = f\frac{l}{d}\frac{V^2}{2g} \quad \text{or} \quad 12.4 = f\frac{600}{\frac{1}{2}}\frac{V^2}{64.4} \quad \text{and} \quad V = \frac{0.815}{\sqrt{f}}$$

But f is unknown and its value depends to some extent on the velocity,



which is also unknown. Using Table VII as a guide, assume a reasonable value of f, say 0.022. Then

$$V = \frac{0.815}{\sqrt{0.022}} = 5.50 \text{ ft./sec.}$$

Referring again to Table VII, it is seen that 0.023 is a more appropriate value of f than 0.022, which was used. Repeating the process with $f = 0.023$,

$$V = \frac{0.815}{\sqrt{0.023}} = 5.38 \text{ ft./sec.}$$

and

$$Q = AV = 0.196 \times 5.38 = 1.05 \text{ c.f.s.}$$

It is never necessary or beneficial to interpolate values of f beyond the two figures given in the table.

Problem 218. The flow in a pipe has an average velocity of 0.8 ft. per sec. Compute the maximum velocity in the pipe (*a*) if $N_R = 4000$, (*b*) if $N_R = 2000$.

219. If the flow in a 3-in. pipe is such that $N_R = N_C/2$, at what distance from the rounded entrance does the velocity distribution become parabolic?

220. An oil having a viscosity of 0.02 slugs per ft. sec. and specific gravity of 0.9 flows through a 2-in. pipe line at one-tenth the critical velocity. If the line is horizontal, what is the pressure difference for points 300 ft. apart?

221. A line of new 6-in. cast-iron pipe delivers water at the rate of 2.0 c.f.s. If the line is horizontal, what is the pressure difference at points 1000 ft. apart? If the flow is from point A to point B, which is 70 ft. higher, what is the difference in pressure, the distance being 1000 ft.?

222. A clean 12-in. cast-iron pipe line shows a drop in energy gradient of 20 ft. in a length of 1500 ft. Compute the quantity flowing in the line.

104. Stanton's Diagram.—The graphical representation in the form shown in Fig. 140 of the relation between values of N_R and the coefficient f in the Darcy formula is frequently called Stanton's diagram because Stanton was among the first to employ a graph in this form.

In this diagram values of the two dimensionless quantities f and N_R are plotted as ordinates and abscissas, respectively, on logarithmic scales. The single straight line at the left gives values of f for laminar flow based upon the Hagen-Poiseuille law. The equation of this line is

$$f = \frac{64}{N_R}$$

It applies to all round pipes and the only data required are those necessary to compute $N_R = \rho Vd/\mu$.

For values of N_R greater than N_C the relationship between f and N_R becomes more complex and is represented graphically by a curve. In the diagram the lower curve gives values of f for smooth tubes such as those made of glass, lead, brass or any drawn metal while the upper curve gives values of f for pipes of steel or cast iron of the quality ordinarily used in commercial work. The curves shown are as drawn by Drew, Koo and McAdams[1] and are based upon a large number of tests by other

[1] DREW, KOO and MCADAMS, The Friction Factor for Clean Round Pipes, *J. Am. Inst. Chem. Eng.*, vol. 28, p. 56, 1932.

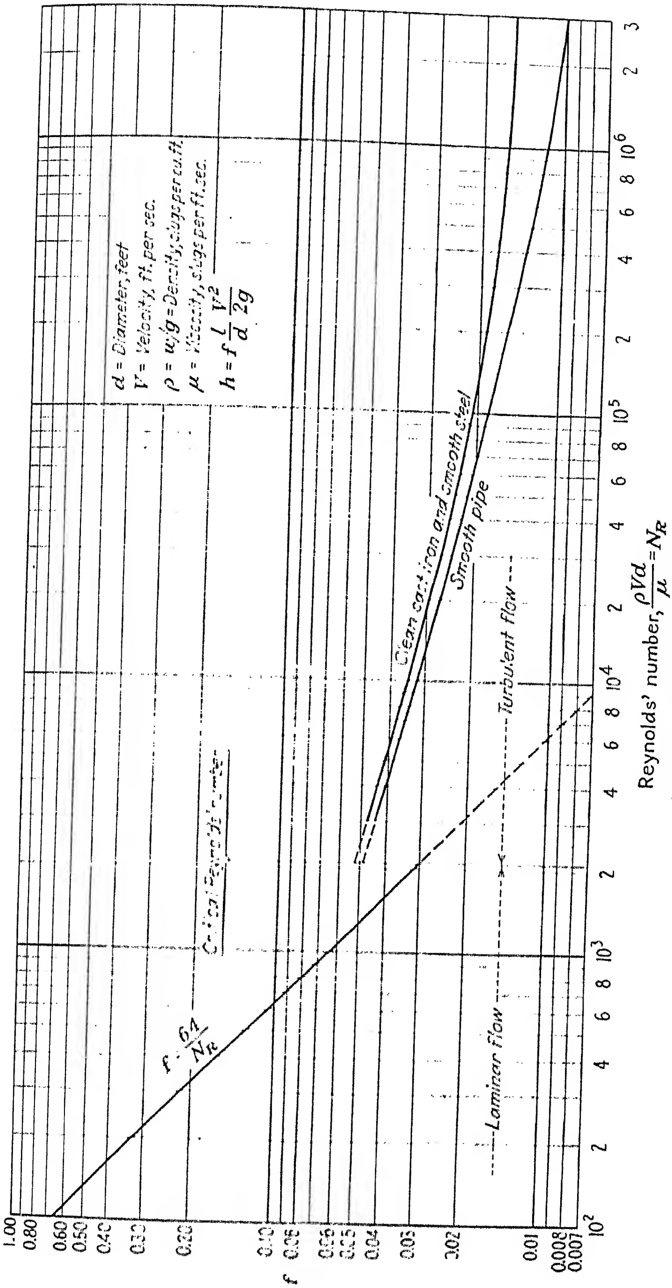


FIG. 140.—Stanton's diagram.

experimenters. The lower curve is in good agreement with that determined by Stanton and others. The points on the curve for high Reynolds' numbers are necessarily obtained by experiments on the flow of air in which V can be very large and μ quite small but the lower curve has been checked with heated water up to $N_R = 1{,}600{,}000$ by Streeter.[1] The values of f in turbulent flow are valid for any fluid and any combination of conditions, with certain limitations in the case of compressible fluids. Within the range of the diagram the values of f for laminar flow are correct except for extremely viscous fluids in very small tubes.[2]

Blasius proposed a formula for loss in smooth pipes which when reduced to the form of the Darcy formula becomes

$$h = \frac{0.316}{\sqrt[4]{N_R}} \frac{l}{d} \frac{V^2}{2g} \tag{21}$$

from which it is seen that the term in the Blasius formula corresponding to f in the Darcy formula is

(22)

If values of f computed from this equation are plotted on Fig. 140, they fit the lower curve well for values of N_R up to 150,000. Equation (22) may then be regarded as an empirical equation for that part of the curve. This fully covers the range of N_R for ordinary hydraulic practice. Beyond this point the equation of the curve is more complicated and the entire length cannot be represented by so simple a function. The equation of Drew, Koo and McAdams

$$f = 0.0056 + 0.500 N_R^{-0.32} \tag{23}$$

fits the curve for smooth pipes very well but is not so convenient in form as Eq. (22).

The position of the curve at the transition from laminar to turbulent flow depends much upon the initial conditions. Either type of flow has a tendency to persist and by exercising great care laminar flow may exist at very high values of N_R. It is

[1] STREETER, V. L., Frictional Resistance in Artificially Roughened Pipes, *Proc. A.S.C.E.*, February, 1935.

[2] PRANDTL and TIETJENS, *op. cit.*, p. 21. This gives lower limits.

always certain that the flow will be laminar below the critical value of $N_R = 2320$ and it is always possible and quite probable that flow will be turbulent for higher values.

Example.—What is the head lost in 1000 ft. of smooth 6-in. steel pipe carrying 0.7 c.f.s. of water at (*a*) 20°C., (*b*) 80°C.?

Solution.—The area of a 6-in. pipe is 0.196 sq. ft. The velocity is

$$V = \frac{Q}{A} = 0.7 \div 0.196 = 3.57 \text{ ft. per sec.}$$

The viscosity of water at 20°C. is 1.008×10^{-2} poises, that is, 0.01008 g. per cm. sec. and, since 1 g. = 6.85×10^{-5} slugs and 1 cm. = 0.0328 ft.,

$$\mu = \frac{0.01008 \times 6.85 \times 10^{-5}}{0.0328} = 2.10 \times 10^{-5} \text{ slugs/ft. sec.}$$

In the same way the viscosity at 80°C. is found to be

$$\mu = 0.744 \times 10^{-5} \text{ slugs/ft. sec.}$$

The Reynolds' numbers at 20°C. and 80°C., considering the change in density with temperature, are then

$$N_R = \frac{Vd\rho}{\mu} = \frac{3.57 \times 0.5 \times 1.943}{2.10 \times 10^{-5}} = 165{,}000 \text{ at } 20°\text{C.}$$

$$N_R = \frac{Vd\rho}{\mu} = \frac{3.57 \times 0.5 \times 1.887}{0.744 \times 10^{-5}} = 453{,}000 \text{ at } 80°\text{C.}$$

and the values of f from Stanton's curve are 0.0195 and 0.0180, respectively. From the Darcy formula the head lost is

$$h = 0.0195 \times \frac{1000}{0.5} \times \frac{(3.57)^2}{64.4} = 7.7 \text{ ft. at } 20°\text{C.}$$

and

$$h = 0.0180 \times \frac{1000}{0.5} \times \frac{(3.57)^2}{64.4} = 7.1 \text{ ft. at } 80°\text{C.}$$

Problem 223. A smooth pipe 6 in. in diameter carries water at a velocity of 8 ft. per sec. Compute the head lost in 1000 ft. of pipe when the temperature of the water is (*a*) 20°C., (*b*) 100°C.

224. A 1-in. pipe carries castor oil at a velocity of 2 ft. per sec. What is the head lost in 100 ft. of pipe (*a*) when the temperature is 20°C., (*b*) when it is 40°C.?

225. A clean 12-in. cast-iron water pipe shows a drop in the hydraulic gradient of 8 ft. per 1000 ft. Compute Q if the temperature is 60°C.

105. Effect of Roughness. The coefficient f for laminar flow is independent, within certain limits, of the roughness of the walls of the pipe. Unless the roughness is so marked that it

constitutes an obstruction to flow or a decrease in the diameter, f may be taken as equal to $64/N_R$. Roughness has a tendency to limit the range of laminar flow, the critical value of N_R being lower for very rough pipes than for smooth ones.

In turbulent flow the roughness of the walls of a pipe has a decided effect on resistance to flow, f, for rough pipes, being much larger than given by the curves of Fig. 140.

Ordinary ideas of roughness as a physical property of a surface cannot be applied directly to the problem of finding f for a pipe and must be supplanted by a notion of hydraulic roughness. Two pipes are said to have the same hydraulic roughness when they have equal values of f for flow at equal values of N_R. They do not necessarily have surfaces of the same texture; on the contrary, rough pipes with wall surfaces of the same texture and different diameter will not have the same f because the relative roughness is different. In order to have the same relative roughness, pipes must have the same values of K/r, where r is the radius and K is some linear measure of absolute roughness. Since roughness cannot be described or characterized by a simple linear dimension, K must be something more complex and at present it is purely hypothetical.

Nikuradse[1] found that pipes with the interior surface covered with sand grains of diameter K' had values of f represented over a wide range of N_R by

$$f = \frac{1}{\left(1.74 + 2 \log_{10} \frac{r}{K'}\right)^2}$$

Some experimenters have sought to characterize roughness by computing from this formula the size of sand grain necessary to produce the f of the experimental pipe in question.[2] The above expression for f does not change with N_R and in general f for a given rough pipe is fairly constant except for values of N_R near the transition from laminar to turbulent flow.

Natural roughness of pipes is usually caused by deterioration such as corrosion, tuberculation or depositing of solids. The change with age is progressive and takes place at a rate depending

[1] NIKURADSE, J., Strömungsgesetze in rauhen Rohren, *V D I*, Forschungsheft No. 361, 1933.

[2] See footnote 1 on p. 206.

upon the material, the fluid and the use and maintenance of the line. It is therefore impossible to state the condition of a rough pipe in terms of material or age. Excessive values of f cannot be attributed to ordinary roughness but must be due to fouling of the pipe by projections or deposits which are so large as to have the effect of reducing the diameter or deforming the passage through which the fluid must flow. It is quite impossible to separate the effects of two kinds of roughness. Many engineers believe that ordinary roughness cannot produce an f exceeding that for commercially smooth pipes at the critical N_R, or 0.054, and on this basis they compute the diameter of a pipe of equivalent capacity having this value of f.

It is usually impossible to separate the effect of surface roughness from that of roughness of the line due to poor joints and poor alignment, which may be more important in some cases than the surface condition. Any condition causing repeated changes in direction, velocity or velocity distribution will greatly increase resistance to flow.

The distribution of velocity in rough pipes does not follow the form shown in Fig. 138 and discussed in Art. 100.

106. Shearing Stress at a Pipe Wall. Resistance to flow in pipes is occasioned by the relative motion of particles of fluid and the actual loss of energy in flow is the work done by internal forces due to relative motion and impact of fluid upon fluid. Since there is no motion at the walls of the pipe, the shear force at the wall cannot be said to perform work. This shear force performs much the same function as the frictional force exerted by a track on the rim of a car wheel. This force is static and does no work although it makes possible the rotation of the wheel and the performing of work at the bearings. The two forces are also similar in that they are a measure of the work performed by internal forces.

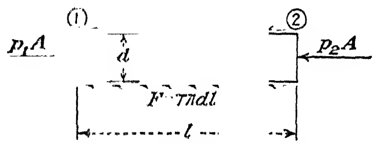


FIG. 141.—Resistance to flow.

The numerical value of the resistance to flow in a pipe can be computed by using the head lost. Figure 141 represents the body of fluid at any time in a pipe of length l and cross section A. For convenience the pipe is taken to be horizontal and of uniform diameter, in which case the head lost is the difference in pressure

heads at points 1 and 2. The pressure difference causing uniform flow from point 1 to point 2 is opposed by a shear force F at the walls. The motion being uniform, the sum of the external forces on the body of fluid is zero or

$$F = p_1A - p_2A \tag{24}$$

From Eq. (15) the pressure drop for laminar flow is

$$p_1 - p_2 = \frac{32\mu lV}{d^2}.$$

Substituting in Eq. (24) and expressing A as $\pi d^2/4$, the resistance to laminar flow is

$$F = 8\pi\mu lV \tag{25}$$

The total shear force on the surface of the cylinder is the unit shear stress τ multiplied by the area πdl or $F = \pi dl\tau$. Substituting this value of F in Eq. (25) and solving for τ, the result is

$$\tau = \frac{8\mu V}{d}. \tag{26}$$

Introducing $N_R = \rho\, Vd/\mu$ in Eq. (26),

$$\tau = \frac{8\rho V^2}{N_R} \tag{27}$$

or in the dimensionless form

$$\frac{\tau}{\rho V^2} = \frac{8}{N_R} = \frac{f}{8} \tag{28}$$

In Stanton's original diagram, he plotted $\tau/\rho V^2$ against N_R and this is common practice in English texts.

In order to get an expression for resistance to turbulent flow, it is necessary to use some empirical formula for pressure drop or lost head. Using the Blasius formula, Eq. (21),

$$p_1 - p_2 = wh = \frac{0.316w}{\sqrt[4]{N_R}}\frac{l}{d}\frac{V^2}{2g}$$

Substituting this value in Eq. (24) and putting $A = \pi d^2/4$,

$$F = \frac{0.316w}{\sqrt[4]{N_R}}\frac{\pi dl}{4}\frac{V^2}{2g} \tag{29}$$

The unit shear stress at the wall is

$$\tau = \frac{F}{\pi dl} = \frac{0.079w}{\sqrt[4]{N_R}} \frac{V^2}{2g} = \frac{0.0395\rho V^2}{\sqrt[4]{N_R}} \tag{30}$$

and with dimensionless terms

$$\frac{\tau}{\rho V^2} = \frac{0.0395}{\sqrt[4]{N_R}} = \frac{f}{8} \tag{31}$$

107. Seventh-root Law for Velocity Distribution in Smooth Pipes.—There are several empirical formulas giving the relation between velocity and position in the cross section of the pipe. Bazin proposed three which were either cubic or biquadratic equations and others are very complicated or are based on false premises. Prandtl and von Kármán, working independently, developed an exponential relation between velocity and position in the cross section founded upon the empirical formula of Blasius for pressure drop. They assume that the velocity distribution at the wall is dependent upon the shear at the wall, that τ is independent of the radius and that the form of the velocity distribution curve does not change with velocity. The latter assumption is expressed mathematically by the statement that the ratio of u to $u_{max.}$ is always the same at a given position in the cross section. Thus, if y' is the actual distance from the wall of the pipe, y'/r is the relative distance in a pipe of radius r and

$$u = u_{max.}\phi\left(\frac{y'}{r}\right) = 1.235V\phi\left(\frac{y'}{r}\right) \tag{32}$$

Supposing $\phi(y'/r)$ to be a simple exponential function, Eq. (32) can be written

$$u = 1.235V \tag{33}$$

In the last article the shear stress at the wall of the pipe, based upon the Blasius formula for pressure drop, was found to be, Eq. (30),

$$\tau = \frac{0.0395\rho V^2}{\sqrt[4]{N_R}} \tag{34}$$

Writing $N_R = \rho Vd/\mu = 2\rho Vr/\mu$, Eq. (34) becomes

$$\tau = \frac{0.0395\rho}{\sqrt[4]{2}} \frac{\mu^{1/4}}{\rho^{1/4} V^{1/4} r^{1/4}} V^2 = \frac{0.0395}{\sqrt[4]{2}} \frac{\rho^{3/4}\mu^{1/4}}{r^{1/4}} V^{7/4}$$

Now substituting the value of V obtained from Eq. (33), the expression for the shearing stress becomes

$$\tau = \frac{0.0395}{\sqrt[4]{2}(1.235)^{7/4}} \frac{\rho^{3/4}\mu^{1/4}u^{7/4}}{r^{1/4}}\left(\frac{r}{y'}\right)^{\frac{7n}{4}} \tag{35}$$

Having assumed that shear stress at the wall is independent of r, the exponent of r in Eq. (35) must be zero, whence

$$\frac{7}{4}n - \frac{1}{4} = 0 \qquad \text{and} \qquad n = \frac{1}{7}$$

Using this value of n in Eq. (33), the exponential equation for velocity at any point at a distance y' from the wall is

$$u = 1.235V\left(\frac{y'}{r}\right)^{1/7} \tag{36}$$

This equation fits the actual turbulent velocity distribution very closely. Thus by placing the origin of coordinates at the wall of the pipe there results a simple exponential formula for velocity distribution which can be used wherever the Blasius formula is applicable.

Problem 226. Compute the velocity at a point 3 in. from the side wall in a smooth 12-in. pipe which carries water at 60°C. and at an average velocity of 2 ft. per sec. What is the maximum velocity in the pipe?

108. Energy Losses Due to Changes in Velocity. When fluid flowing in a pipe is forced to undergo any change in velocity or velocity distribution, there is a loss of energy or head, that is, energy in the forms included in the Bernoulli constant is transformed to heat. At the entrance to a pipe line, at enlargements, contractions or obstructions, and at bends the change in velocity or velocity distribution entails losses which in some cases comprise a relatively large part of the total loss and cause a considerable part of the resistance to flow.

The amount of such losses can be approximately determined in some cases by theoretical means, but on the whole their computation depends upon certain experimentally determined coefficients.

Sudden enlargement offers the best opportunity to determine a loss by theoretical means. In the sudden enlargement shown in Fig. 142 the stream initially has a cross section A_1 and is

discharged into a pipe of cross section A_2. The transformation in size of the stream requires some distance and the face of the expanding stream is a surface of discontinuity outside of which is eddying fluid not taking part in the general downstream motion. The pressure in the jet at the beginning of the large pipe is the same as that in the surrounding fluid. Considering the pressure forces on the body of fluid $CDEF$, there is a total force of p_1A_2 acting toward the right on face CD and a force p_2A_2 exerted toward the left on face EF. If the shear force along the wall of the pipe is neglected, the net effective force accelerating the body $CDEF$ is $p_1A_2 - p_2A_2$. In passing through the space $CDEF$, a quantity of Q c.f.s. has its velocity changed from V_1 to V_2. Equating the effective force to the rate of change of momentum,

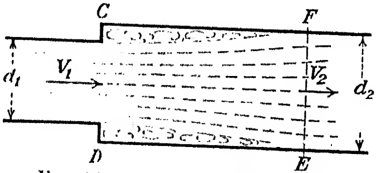


FIG. 142. Sudden enlargement.

$$(p_1 - p_2)A_2 = \frac{Qw}{g}(V_2 - V_1) = \frac{wA_2V_2}{g}(V_2 - V_1)$$

and

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{V_2}{g}(V_1 - V_2) \tag{37}$$

Writing Bernoulli's equation between points 1 and 2 and correcting for the loss h_e gives

$$\frac{V_1^2}{2g} + \frac{p_1}{w} = \frac{V_2^2}{2g} + \frac{p_2}{w} + h_e$$

from which the loss is

$$h_e = \left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right) - \left(\frac{p_2}{w} - \frac{p_1}{w}\right) \tag{38}$$

Substituting the value of the latter term as given by Eq. (37), the loss is found to be

$$h_e = \left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right) - \frac{V_2}{g}(V_1 - V_2)$$

$$h_e = \frac{(V_1 - V_2)^2}{2g} \tag{39}$$

Equation (39) is known as Borda's formula. It is based entirely on theoretical considerations, but for water it is well supported by the experimental work of W. H. Archer,[1] who developed the formula

$$h_e = 1.098\frac{(V_1 - V_2)^{1.919}}{2g} \tag{40}$$

It is often desirable to express this loss in the form $h_e = K_e\dfrac{V_1^2}{2g}$ in which K_e is a coefficient varying in value with the ratio of velocities, diameters or areas. Equating this expression for h_e to that in Eq. (39) and solving for K_e,

$$K_e = \frac{(V_1 - V_2)^2}{V_1^2} = \left(1 - \frac{A_1}{A_2}\right)^2 = \left[1 - \left(\frac{d_1}{d_2}\right)^2\right]^2 \tag{41}$$

and

$$h_e = K_e\frac{V_1^2}{2g} = \left[1 - \left(\frac{d_1}{d_2}\right)^2\right]^2\frac{V_1^2}{2g} \tag{42}$$

Sudden contraction produces losses that can be completely determined only with the help of experimental coefficients.

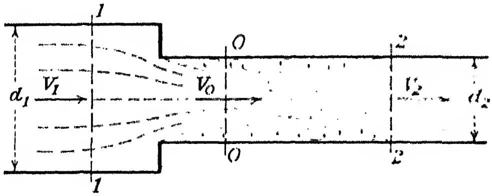


FIG. 143. Sudden contraction.

Figure 143 represents a pipe in which the area is suddenly reduced from A_1 to A_2. When fluid is made to pass over a surface having such a sharp curvature that it cannot follow it, as around the sharp corner in this case, the stream leaves the surface and is contracted. As the fluid rounds the sharp corner of the sudden contraction, the cross section is reduced or contracted from A_2 to A_0 and $A_0 = C_cA_2$, in which C_c is a coefficient of contraction. This coefficient changes with the ratio A_2/A_1, being larger for large values of the ratio and reaching a minimum of about 0.62 for water when A_1 is very large. Downstream from plane 00

[1] ARCHER, W. H., Experimental Determination of Loss of Head Due to Sudden Enlargement in Circular Pipes, *Trans. A.S.C.E.*, vol. 76, p. 999, 1913.

the stream expands to fill the pipe again. The loss in this expansion is far larger than the loss in the contraction just upstream. Considering the latter to be negligible, the loss may be determined from Eq. (39), which, when applied here, becomes

$$h_c = \frac{(V_0 - V_2)^2}{2g} \tag{43}$$

Expressing the loss in contraction in the form $h_c = K_c\dfrac{V_2^2}{2g}$, equating this to h_c from Eq. (43) and solving for K_c,

$$K_c = \frac{(V_0 - V_2)^2}{V_2^2} = \left(\frac{A_2}{A_0} - 1\right)^2 = \left(\frac{1}{C_c} - 1\right)^2 \tag{44}$$

and

$$h_c = K_c\frac{V_2^2}{2g} = \left(\frac{1}{C_c} - 1\right)^2\frac{V_2^2}{2g} \tag{45}$$

Weisbach[1] gives the following values of C_c for sudden contraction with water:

A_2/A	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C_c	0.624	0.632	0.643	0.659	0.681	0.712	0.755	0.813	0.892	1.00

The loss as given by Eq. (45) is only approximate as it is based upon theory which neglects the loss upstream from section 00 of Fig. 143. Furthermore, experiments show that there is a slight variation of K_c with V.

The loss at entrance to a pipe line depends upon the form of the opening. The sharp-cornered entrance of Fig. 144 is a special case of sudden contraction. The total loss is about $0.48V^2/2g$ and is usually taken to be $0.5V^2/2g$. Commonly used values for the loss computed from Eq. (19) of Chap. XI are shown in Fig. 144. The rounded entrance causes a very small loss while the re-entrant pipe causes a large loss.

Bends, tees, valves and other fittings or obstructions in a line of pipe cause losses which depend in amount upon the construction used. Any such loss may be stated in the form $h = K\dfrac{V^2}{2g}$. In

[1] WEISBACH, JULIUS, "Die Experimental-Hydraulik," p. 133, J. S. Engelhardt, Freiberg, 1855.

work involving a large number of such fittings, it is convenient to adjust for such losses by increasing the length of pipe in the Darcy formula by a length which it is estimated would cause the same loss as the fitting.[1] These equivalent lengths are based on the assumption that the pipe is in good condition. For pipes with a different value of f the equivalent length may be adjusted according to the ratios $l_1/l_2 = f_2/f_1$.

Knowledge of the subject of minor losses in pipes for all fluids appears to be in an unsatisfactory state. All the values given here may be considered approximations. Any enlargement,

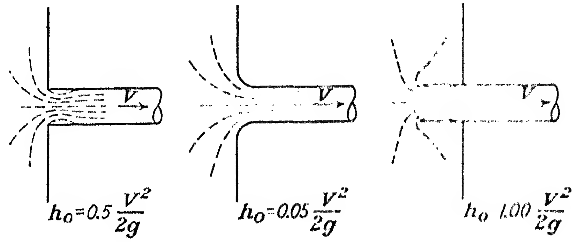


FIG. 144.—Loss at entrance to a pipe line.

contraction or other fitting disarranges the velocity distribution, and for that reason the losses caused by them are not confined to their immediate vicinity but extend downstream until the velocity distribution is again normal. Bends always set up a rotation in a pipe which may have an important effect on loss in the line downstream from the bend.

As shown in Art. 101 of this chapter, the energy involved in laminar flow is much greater for a given average velocity. Laminar flow requires a much greater distance downstream from a disturbance for the velocity distribution to return to normal and the rotation induced by a bend is probably more important than for turbulent motion. It is to be expected then that the coefficient K in the expression $h = K\dfrac{V^2}{2g}$ is somewhat larger for laminar flow than those given here for turbulent flow.

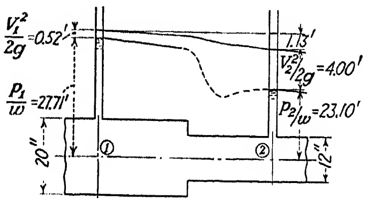
[1] For equivalent lengths for fittings, see D. E. Foster, Effect of Fittings on Flow of Fluids through Pipe Lines, *Trans. A.S.M.E.*, vol. 42, p. 647, 1920; and W. A. Thomas, "Resistance of Fittings Chart," p. 203, The Valve World, Crane Co., Chicago, November, 1932.

Example.—The pressure in a pipe line changes from 12 lb. per sq. in. to 10 lb. per sq. in. where the diameter is changed suddenly from 20 to 12 in. Compute the quantity flowing in the line.

Solution.—The pressure heads at points 1 and 2 are

$$\frac{p_1}{\;} = 12 \quad 0.433 = 27.71 \text{ ft.}$$

and $p_2/w = 10 \div 0.433 = 23.1$ ft. These are plotted in the figure. The loss as given by Eq. (45), taking Weisbach's values of C_c for $A_2/A_1 = 0.36$, is



$$h_c = \left(\frac{1}{C_c} - 1\right)^2 \frac{V_2^2}{2g}$$

$$= \left(\frac{1}{0.653} - 1\right)^2 \frac{V_2^2}{2g} = 0.283\frac{V_2^2}{2g}$$

Bernoulli's equation between 1 and 2, corrected for this loss, is

$$\frac{V_1^2}{2g} + 27.71 = \frac{V_2^2}{2g} + 23.10 + 0.283\frac{V_2^2}{2g}$$

After finding from $A_1V_1 = A_2V_2$ that $\frac{V_1^2}{2g} = 0.130\frac{V_2^2}{2g}$,

$$4.61 = \frac{V_2^2}{2g} + 0.283\frac{V_2^2}{2g} - 0.130\frac{V_2^2}{2g} = 1.153\frac{V_2^2}{2g}$$

from which $V_2^2/2g = 4.00$ and $V_2 = 16.04$ ft. per sec. The discharge is $Q_2 = A_2V_2 = 0.785 \times 16.04 = 12.59$ c.f.s. The drop in the energy gradient is the lost head, in this case $0.283V_2^2/2g = 1.13$ ft., and the initial velocity head is $0.130V_1^2/2g = 0.52$ ft.

Problem 227. A water pipe carrying 8 c.f.s. is enlarged abruptly from 12 to 18 in. in diameter. What is the change in the energy grade line and in the hydraulic grade line?

228. A water pipe carrying 8 c.f.s. is reduced abruptly from 18 to 12 in. in diameter. Compute the change in energy and hydraulic grade lines.

229. A 12-in. water pipe connected to a reservoir has a sharp-cornered entrance. The hydraulic grade line just downstream from the entrance is 1.5 ft. below the surface of the reservoir. Compute the quantity flowing in the pipe. If the line had a carefully rounded entrance, what would be the drop in hydraulic grade line with the same quantity flowing?

109. Hydraulic and Energy Gradient for Nonuniform Flow.—In the case shown in Fig. 135 and discussed in Art. 97 the hydraulic grade line is affected only by the head lost in uniform flow. When the flow is nonuniform as in a line having contractions and enlargements, the changes in velocity and velocity head and the resulting losses complicate the hydraulic grade line. Velocity

being fixed by the conditions of the continuity equation, and elevation being controlled by the position of the pipe, all changes in velocity head and all losses must be reflected in the pressure-head term of the Bernoulli constant. The fall in the pressure gradient in this case is then due to the loss of head and the creation of velocity head.

In drawing the grade line shown in Fig. 145 it is assumed that the losses and transformation of energy take place in a short distance as compared with the length of straight pipe, that the gradient for the straight portion is a straight line unaffected by

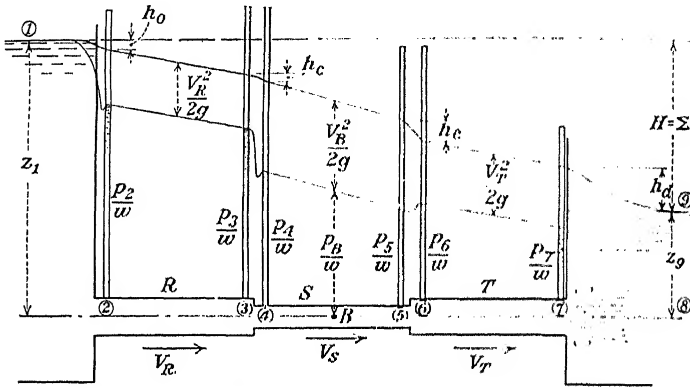


FIG. 145.—Hydraulic grade line for nonuniform flow.

the disarrangement of velocity distribution at the changes, and that the velocity head $V^2/2g$ represents the average kinetic energy content per pound of fluid. In spite of the fact that the two latter assumptions are never strictly correct, the grade line is helpful in studying the flow.

The pressure drop at the entrance can be determined by Bernoulli's equation for points 1 and 2, whence

$$z_1 = \frac{V_2^2}{2g} + \frac{p_2}{w} + h_0 \text{ (loss at entrance)}$$

the pressure head at point 2 being

$$\frac{p_2}{w} = z_1 - \left(\frac{V_2^2}{2g} + h_0\right) \tag{46}$$

It is seen from this equation that the drop in grade line is

$$z_1 - \frac{p_2}{w} = \frac{V_2^2}{2g} + h_0$$

It should be noted that it does not consist of the loss alone but is affected by both loss and change in velocity head.

Applying the same method, it can be shown that the drop in grade line between points 3 and 4 is the increase in velocity head plus the loss, that is,

$$\frac{p_3}{w} - \frac{p_4}{w} = \left(\frac{V_4^2}{2g} - \frac{V_3^2}{2g}\right) + h_c \tag{47}$$

and it can likewise be shown that there is a rise in pressure head at the enlargement between points 5 and 6 of

$$\frac{p_6}{w} - \frac{p_5}{w} = \left(\frac{V_5^2}{2g} - \frac{V_6^2}{2g}\right) - h_e \tag{48}$$

owing to the partial transformation of the higher velocity head at 5 to pressure head at 6. The value of h_e is given by Eqs. (39), (40) or (42) and h_c can be computed from Eq. (43) or (45).

When a pipe discharges into a reservoir of liquid as in Fig. 145, most of the velocity head is lost. This is a special case of sudden enlargement. Writing the Bernoulli equation between points 7 and 9 and computing the loss from the Archer formula, Eq. (40), which in this case becomes

$$h_e' = 1.098 \cdot \frac{V_T^{1.919}}{2g} \tag{49}$$

it is found that $\frac{p_7}{w}$ is less than z_9. The quantity $z_9 - \frac{p_7}{w}$ represents that part of $V_T^2/2g$ which is not lost when the stream expands in the reservoir. It is quite common practice to assume that the loss is just $V_T^2/2g$, in which case $p_7/w = z_9$.

If the quantity flowing is known, the pressure anywhere in the line can be found by using the Bernoulli equation between point 1 and the point in question and including all losses.

Bernoulli's equation between points 1 and 9 takes the form

$$z_1 = z_9 + (\text{all losses between 1 and 9})$$

from which

$$z_1 - z_9 = h_0 + h_R + h_c + h_S + h_e + h_T + h_e' = H \tag{50}$$

or

$$H = h_0 + f_R\frac{l_R}{d_R}\frac{V_R^2}{2g} + K_c\frac{V_S^2}{2g} + f_S\frac{l_S}{d_S}\frac{V_S^2}{2g} + K_e\frac{V_S^2}{2g} + \\ + f_T\frac{l_T}{d_T}\frac{V_T^2}{2g} + K_e'\frac{V_T^2}{2g} \tag{51}$$

At first glance this equation appears quite formidable. The velocities are related through the continuity equation but some of the coefficients are dependent upon the unknown velocities. In solving for any velocity or Q it is necessary to use a series of approximations starting with assumed values of f which can be corrected when making the second solution. With Q given, the solution for H is direct.

The upper line in Fig. 145 is plotted by adding the velocity head to the elevation of the pressure gradient, its elevation at any point B being $\frac{p_B}{w} + \frac{V_B^2}{2g}$. This line is the energy gradient and its elevation above datum represents the total energy referred to datum. The vertical distance from the energy line at any point to the level of point 1 represents the energy lost and it follows that the line must always slope downward in the direction of flow.

The slope of the hydraulic gradient is the hydraulic slope of the pipe, independent of the position of the pipe. The pipe in Fig. 145 could have been in a position higher, lower or inclined, within limits, without change in the hydraulic grade line or slope.

When a pipe line contains a large number of valves, bends and changes in size and where the straight runs are comparatively short, the hydraulic grade line cannot be accurately drawn. The velocity distribution is never ideal and the conditions under which experimental values of f are determined are not even approximated. Such problems require the exercise of considerable judgment as accurate computation of losses is not possible.

110. Minor Losses Neglected.—In many pipe lines the fluid flows through comparatively great distances between disturbances such as those caused by obstructions, contractions, enlargements or bends. If the distances are sufficiently large the minor losses due to these disturbances are relatively unimportant and may be dropped from such an equation as Eq. (51), which then becomes

$$H = f_R \frac{l_R}{d_R} \frac{V_R^2}{2g} + f_S \frac{l_S}{d_S} \frac{V_S^2}{2g} + f_T \frac{l_T}{d_T} \frac{V_T^2}{2g} \qquad (52)$$

The uncertainty in choosing values of f is often such that there is no real refinement of results to be had by using the minor

losses. The length of pipe for which such losses are negligible depends upon the conditions at hand and the accuracy desired. If the flow is turbulent, the minor losses have relatively little effect on discharge when the undisturbed lengths are 1000 diameters or more. In solving for V and Q the errors introduced will be small but, when H is computed from a known Q, the percentage of error will be much larger.

For the pipe line shown in Fig. 146 the Bernoulli equation from the free surface to the end of the pipe, corrected for losses and referred to a datum plane through the outlet, is

$$H = \frac{V^2}{2g} + 0.5\frac{V^2}{2g} + f\frac{l}{d}\frac{V^2}{2g} \tag{53}$$

Neglecting the velocity head and entrance loss, this becomes

$$H = f\frac{l}{d}\frac{V^2}{2g} \qquad \text{or} \qquad V = \sqrt{\frac{2gHd}{fl}} \tag{54}$$

and the hydraulic grade line is merely a line assumed to be drawn straight from the free surface to the outlet.

Problem 230. In Fig. 145 the lengths of the three pipes are $l_R = 300$ ft., $l_S = 100$ ft. and $l_T = 450$ ft., and the diameters are 18, 12 and 18 in., respectively. The quantity flowing is 10 c.f.s. and the pipes are new cast iron. Compute the total head H and plot the hydraulic and energy grade lines.

231. In Fig. 145 the lengths of the three pipes are $l_R = 500$ ft., $l_S = 1200$ ft. and $l_T = 900$ ft., and the diameters are 12, 18 and 12 in. respectively. The material is old cast iron and the total head is 30 ft. Compute the flow of water (*a*) considering all losses, (*b*) neglecting minor losses and velocity changes.

111. Other Hydraulic Gradients.—If the pipe is of uniform condition or roughness and if the slope of the pipe is such that axial lengths do not differ much from the horizontal distances, the hydraulic gradient is essentially straight.

Thus the hydraulic grade line for the pipe of Fig. 146, neglecting minor losses, is a line from the free surface to the outlet. If minor losses are considered, the grade line drops a distance $\frac{V^2}{2g} + h_0$ at the entrance end.

The pressure head at such points as B and D is indicated by the vertical distance of the pipe below the hydraulic gradient and

it follows that, when the pipe is above the hydraulic gradient, the pressure head is less than atmospheric by an amount equal to the difference in elevation. This difference is limited theoretically to the difference between the atmospheric pressure head and the vapor pressure, that is, $\frac{p_a}{w} - \frac{p_v}{w}$. For practical reasons the negative head is further limited because at such low

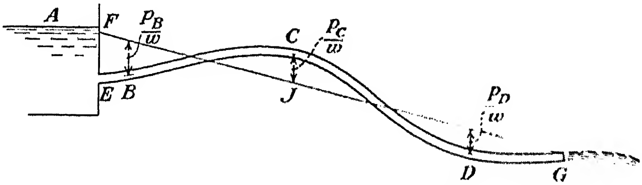


FIG. 146.—Hydraulic grade line and profile of a pipe line.

pressures there is a tendency for air entrained in a liquid to separate and collect at a crest such as point C. Even in lines entirely under pressure this accumulation of air may become troublesome, for the pipe becomes "air bound" and its capacity is greatly reduced unless provision is made to remove the air. A pipe in the position shown in Fig. 146 must be structurally

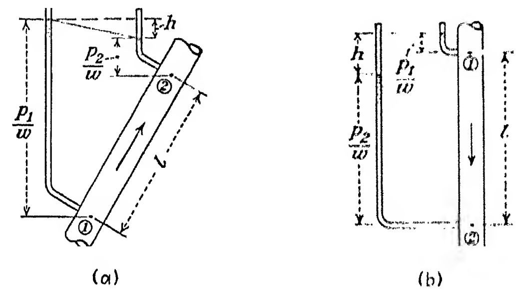


FIG. 147.—Pipes on steep inclines.

capable of resisting a load from the outside. Other pipes often collapse when accidentally subjected to negative pressure.

In the case of the vertical or sharply inclined pipes of Fig. 147, it is impractical to draw a true hydraulic gradient. Since it is merely a device to assist in the study of pipe flow, the user is free to adopt any workable convention as, for example, in Fig. 147*a*. In this case the hydraulic slope of the pipe is h/l. This, however, is not the slope of the grade line as drawn. In the

vertical pipe of Fig. 147*b* the liquid always stands higher in the upstream tube by the amount of the lost head and the hydraulic slope is again h/l.

Problem 232. Sketch the hydraulic grade line for Fig. 146, (*a*) with a nozzle on the end of the line, (*b*) with a vent to the atmosphere at C, (*c*) with the pipe closed at G, (*d*) with the pipe cut off at D.

233. The water pipe in Fig. 146 is 800 ft. long and 12 in. in diameter and $f = 0.02$. Point G is 30 ft. below A and C is 6 ft. below A and 300 ft. from E. Compute the discharge and the pressure at C. Correct for all losses.

112. Divided Flow in Pipes.—In such a system of pipes as shown in Fig. 148 the liquid flowing from reservoir A to reservoir

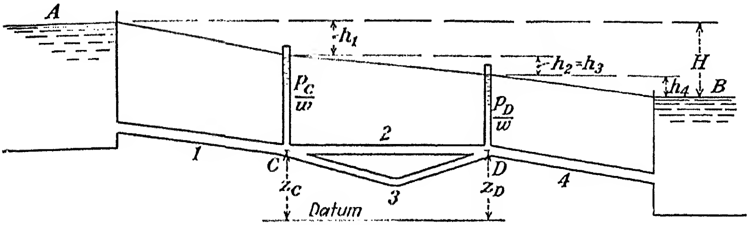


FIG. 148. Divided pipe flow.

B is divided between pipes 2 and 3. The drop in the hydraulic grade line in either pipe is

$$\left(\frac{p_C}{w} + z_C\right) - \left(\frac{p_D}{w} + z_D\right) \qquad h_2 = h_3$$

and the flow must divide itself to satisfy this condition. In doing this the flow in the two pipes from C to D is analogous to the flow of electricity in parallel conductors over which the drop in potential must be equal. Neglecting minor losses, the conditions shown establish the following equations:

$$\begin{aligned} h_2 &= h_3 \\ h_1 + h_2 + h_4 &= H \\ Q_1 &= Q_4 \\ Q_1 &= Q_2 + Q_3 \end{aligned} \tag{55}$$

Each h and Q may be expressed in terms of a corresponding velocity and there are then four equations in V_1, V_2, V_3, V_4, the first two of which contain values of f depending to some extent on the unknown velocities. A direct solution may then

be both difficult and futile and the trial solution is indicated. One convenient method of solving for Q when H is known is to assume a value of h_1 and compute Q_1, Q_4, h_4, h_2, h_3, Q_2, Q_3 in order. If the computed values of Q satisfy the last of Eqs. (55), they are correct; otherwise another trial is necessary. On the next trial the solution may be started with an assumed value of h_1 or Q_1, using the first results as a guide.

In the system shown in Fig. 149 the flow in pipes 1 and 3 is evidently toward the right. If the elevation of C and the elevation of the free surfaces in the reservoirs are given, there is

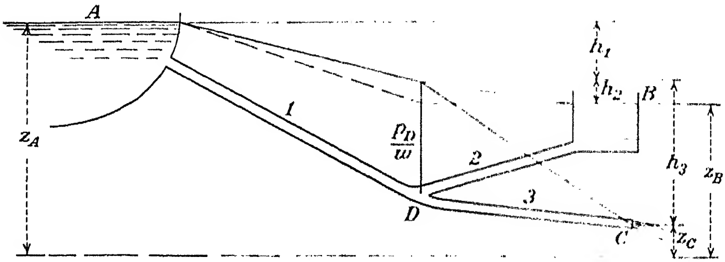


FIG. 149.—Branching pipes.

often no indication of the direction of flow in pipe 2. The direction can be quickly determined by assuming no flow in pipe 2, thus placing the hydraulic grade line (dotted) at the junction D at the same level as the free surface in B. Q_1 and Q_3 can then be computed. If Q_1 is greater than Q_3, it is evident that the flow is into B. It is then known that the grade line at D is above B and the following conditions are to be fulfilled:

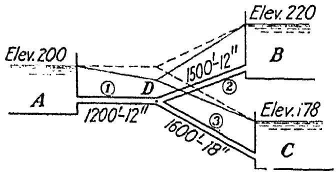
$$\left.\begin{aligned} h_1 + h_2 &= z_A - z_B \\ h_1 + h_3 &= z_A - z_C \\ Q_2 + Q_3 &= Q_1 \end{aligned}\right\} \tag{56}$$

These equations can be satisfied by trial, one method being to assume h_1, compute h_2, h_3, Q_1, Q_2, Q_3 and check the values of Q by the last equation.

Example.—In the figure reservoirs A, B and C are connected by pipes 1 and 2, which are clean cast iron, and pipe 3, which is old cast iron. Find the quantity flowing in each pipe.

Solution.—The direction of flow is at first unknown and it is therefore impossible to set up equations for the system. If it is first supposed that there is no flow in pipe 1, the grade lines are as shown in dotted lines. Then

$h_2 = 20$ ft., $h_3 = 22$ ft., and Q_2 and Q_3 can be computed. For this purpose it is convenient to write the Darcy formula as $V = \sqrt{2ghd/lf}$. Considering only the friction losses and assuming f_2 to be 0.020,



$$V_2 = \sqrt{\frac{64.4 \times 20}{1500 \times 0.020}} = 6.55 \text{ ft./sec.}$$

and

$$Q_2 = A_2V_2 = 0.785 \times 6.55 = 5.15 \text{ c.f.s.}$$

Likewise, taking $f_3 = 0.038$ from Table VII,

$$V_3 = \sqrt{\frac{64.4 \times 22 \times 1.5}{1600 \times 0.038}} = 5.91 \text{ ft./sec.}$$

and

$$Q_3 = A_3V_3 = 1.77 \times 5.91 = 10.46 \text{ c.f.s.}$$

It is obvious that these values of Q_2 and Q_3 are incompatible. There must be flow toward the right in pipe 1, which means that the grade line at D is lower than elevation 200, the new system of grade lines being as shown by the full lines. It is now possible to write the equations for the system, which are

$$h_2 - h_1 = 20, \qquad h_1 + h_3 = 22, \qquad Q_3 = Q_1 + Q_2$$

These can be solved by trial. With the grade line lowered at D, it is apparent that Q_2 must be greater than 5.15 c.f.s. and that Q_3 must be less than 10.46 c.f.s. Hence the flow in pipe 1 must be less than

$$10.46 - 5.15 = 5.31 \text{ c.f.s.}$$

As a first trial, assume that Q_1 is, say, 3 c.f.s. Then

$$V_1 = \frac{Q_1}{A_1} = 3 \div 0.785 = 3.82 \text{ ft./sec.}$$

and, taking f from the table,

$$h_1 = 0.22 \cdot \frac{1200}{1} \cdot \frac{(3.82)^2}{64.4} = 6.0 \text{ ft.}$$

Now $h_2 = 20 + 6 = 26$ and $h_3 = 22 - 6 = 16$, from which

$$V_2 = \sqrt{\frac{64.4 \times 26 \times 1}{1500 \times 0.021}} = 7.28 \text{ ft./sec.}$$

so that $Q_2 = 5.71$ c.f.s. and

$$V_3 = \sqrt{\frac{64.4 \times 16 \times 1.5}{1600 \times 0.038}} = 5.04 \text{ ft./sec.}$$

so that $Q_3 = 8.92$ c.f.s. Now the assumed Q_1 plus the computed Q_2 is $3 + 5.71 = 8.71$ c.f.s. as compared to 8.92 c.f.s., the computed value of Q_3.

This is considered good agreement and well within the limits of error in choosing values of f for a pipe. Then the discharge can be taken as $Q_1 = 3$, $Q_2 = 5.8$ and $Q_3 = 8.8$. If good agreement is not obtained, another trial is necessary. The neglect of the small losses at entrance and exit has little effect on the results or on the accuracy in this problem.

113. Equivalent Pipes.—Many systems are so complicated that it becomes convenient to substitute a pipe of equivalent capacity for two or more pipes of the system. The new pipe can be made of equivalent capacity for one set of arbitrary conditions which should approximate the conditions under which the system is operating. For example, pipes 2 and 3 of Fig. 148 could be replaced by an equivalent pipe having an arbitrary length l between lengths l_2 and l_3 and a capacity $Q = Q_2 + Q_3$ with the same loss of head as pipes 2 and 3. The value of h should be about the same as in service, and a value of f about the same as for pipes 2 and 3 should be used. Then, after computing Q_2 and Q_3 from the arbitrary h,

$$Q = Q_2 + Q_3, \qquad V = \frac{Q}{A} = \frac{4Q}{\pi d^2}$$

$$h = f\frac{l}{d}\frac{V^2}{2g} = f\frac{l}{d}\frac{16Q^2}{2g\pi^2 d^4}$$

and the diameter of the equivalent pipe is

$$d = \sqrt[5]{\frac{8flQ^2}{\pi^2 gh}} \tag{57}$$

By repeating this or similar operations many systems of pipes either in series or parallel can be reduced to a single equivalent pipe.

Problem 234. A test of a 10-in. pipe shows that f 0.32, which is excessive for a clear line. Assuming that legitimate roughness cannot produce a value of f more than 0.05, compute the diameter of an equivalent pipe of this roughness.

235. In Fig. 148 the lengths of the pipes are $l_1 = 2000$ ft., $l_2 = 1800$ ft., $l_3 = 1000$ ft., $l_4 = 2400$ ft., and the diameters are 18, 12, 12 and 24 in., respectively. Letting $Q_1 = 12$ c.f.s. of water and taking f from Table VII, compute H. The pipes are clean cast iron.

236. In Fig. 149 the pipe lengths are $l_1 = 2000$ ft., $l_2 = 1000$ ft., $l_3 = 1500$ ft. The diameters are 12, 12 and 18 in., respectively, and the elevations of A, B and C are 200, 180 and 150 ft., respectively. Compute Q_3, using f for clean cast-iron water pipe.

237. In Fig. 149 take diameters and lengths as in Prob. 236. A and C are at elevations 200 and 170, respectively, and $Q_3 = 10$ c.f.s. of water. Find the elevation of B, taking $f = 0.02$ for all pipes.

114. Flow of Compressible Fluids.—In Art. 104 it is stated with some qualification that Stanton's curve can be applied to both liquid and gaseous fluids and that much of the curve is in fact determined by experiments on flow of air. It can be further stated that the methods of this chapter are in general applicable to all fluids within certain limits. The flow of compressible fluids may be entirely similar to the flow of a fluid at some section of a pipe but quite different in respect to the variation along the length of the pipe. The absorption by the fluid of all or part of the energy lost and the changes in temperature, pressure, density and viscosity must be given consideration according to the principles of thermodynamics. These problems, which become somewhat involved, are discussed in Chap. XIV. Acceptable results following the methods of this chapter may often be obtained by dividing a line of pipe into reaches over which the pressure drop is small as compared to the absolute pressure and the Reynolds' number nearly constant. There is no safe general rule for telling when this method can be safely applied and the results of such work should be frequently checked against the more exact theory of Chap. XIV.

The discussion in this chapter of minor losses applies particularly to flow of liquids and should not be applied to compressible fluids where changes in velocity are great.

115. Noncircular Pipes. The work of this chapter thus far has referred to the flow of fluids in pipes of circular cross section. It can be applied with fair results, however, to turbulent flow in noncircular tubes and annular spaces. The term d in N_R or in the Darcy formula must be replaced by a new term having the same dimension as d, for which purpose the hydraulic radius is introduced. The hydraulic radius of any cross section is

$$R = \frac{\text{area of cross section}}{\text{wetted perimeter of cross section}}$$

In the case of a round pipe

$$R = \frac{\pi d^2/4}{\pi d} = \frac{d}{4} \qquad \text{or} \qquad d = 4R \tag{58}$$

Substituting this value for d, the Reynolds' number becomes

$$N_R = \frac{4VR\rho}{\mu} \tag{59}$$

and the Darcy formula is

$$h = \frac{f}{4}\frac{l}{R}\frac{V^2}{2g} \tag{60}$$

Equations (59) and (60) used in connection with the Stanton diagram give fair results for all ordinary shapes if the flow is turbulent.

When the flow is laminar these equations cannot be used except for shapes nearly circular or square. The difference in results for the two types of flow is attributed to the great difference in

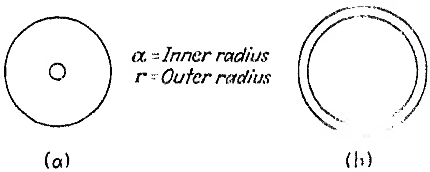


FIG. 150.—(a) Annular space with large ratio r/a. (b) Annular space with small ratio r/a.

velocity distribution. The error introduced by using the hydraulic radius is especially large for laminar flow in annular spaces such as those shown in Fig. 150. The equation for laminar flow given by Lamb[1] for the quantity flowing through a tube of this form is

$$Q = \frac{\pi}{8\mu}\frac{(p_1 - p_2)}{l}\, r^4 - a^4 - \frac{(r^2 - a^2)^2}{\log_e (r/a)} \tag{61}$$

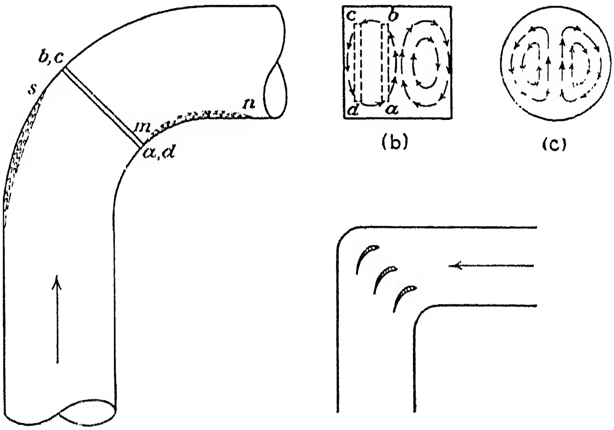
When $r/a = 10$ this equation gives a correct value of Q only slightly more than half of that for a full circle as computed from Eq. (16) of Chap. VIII. The hydraulic radius method in this case would give a value of Q which is about 80 per cent too large.

When the proportions are as shown in Fig. 150b, the flow can be computed by Eq. (61) or as between two parallel plates of

[1] LAMB, H., "Hydrodynamics," 6th ed., p. 587, Cambridge University Press. 1932.

width $2\pi\frac{r+a}{2}$ with a space between of thickness $r - a$. The equations for this latter case are given in Chap. XVI.

116. Flow in Pipe Bends.—When fluid is made to flow around a bend in a pipe, the centrifugal force which is introduced produces an excess of pressure on the outer surface of the bend. If it should be supposed that the flow is laminar and that each lamina remains parallel to the axis of the bend, then the outer laminas would have to acquire more velocity. But this increase in velocity would necessitate a reduction in pressure at the out-



(a) (d)

FIG. 151. Flow in bends.

side of the bend which is quite incompatible with the above idea of an increase due to centrifugal force. The supposition that the motion around the bend is a mere speeding up of outer particles or retardation of the inner ones must therefore be abandoned.

Consider the prisms *ab* and *dc* in the square cross section of Fig. 151*b*. Prism *ab* has the higher velocity because it is less affected by the sides than *dc*, and accordingly the centrifugal force is greatest on *ab*. This builds up a pressure difference such that $p_b > p_c$ and $p_d > p_a$ and the motion indicated by the arrows is induced, with a similar motion in the opposite direction in the other half of the section. This motion is also present in round pipes, as shown in Fig. 151*c*, and exists in both laminar and turbulent flow. The induced or secondary motion is superim-

posed upon the original motion and it can be shown that the resultant motion may involve a smaller total energy content than when the velocity is merely assumed to increase outside the axis and decrease within. The pressure increase at the outside and decrease at the inside may cause eddies in the regions *mn* and *rs*, the latter being eliminated as soon as the secondary motion or vortex pair is strong enough.[1]

When this secondary motion is once established, it tends to persist for a considerable distance downstream so that the losses caused by a bend are not confined to the bend itself. In experimental work with air the secondary motion is eliminated or minimized by providing vanes at the bend as shown in Fig. 151*d*.

General Problems

238. A 30-in. water pipe carries 20 c.f.s. At point *B* the pressure is 25.5 lb. per sq. in. gage and the elevation is 120 ft. At point *C*, which is 5000 ft. from *B*, the pressure is 45.2 lb. per sq. in. absolute and the elevation is 100 ft. Compute f in the Darcy formula.

239. A liquid having a viscosity of 2 poises and a weight of 60 lb. per cu. ft. is pumped through a smooth 6-in. pipe at the rate of 0.1 c.f.s. What is the difference in pressure at two points on the same level and 6000 ft. apart? What is the actual kinetic energy per pound of fluid flowing?

240. A pump at an elevation of 60 ft. is connected to a reservoir at an elevation of 100 ft. by 4500 ft. of old 18-in. cast-iron pipe. At a point in this line 1500 ft. from the reservoir an old 12-in. pipe is connected. It is 1200 ft. long and leads to a point at elevation 85 where it discharges at the rate of 6 c.f.s. Taking f as 0.032 for all pipes, compute the pressure that must be maintained by the pump. The fluid is water.

241. Water is pumped through a smooth 6-in. pipe line 1200 ft. long at the rate of 100 g.p.m. What is the head required if the temperature is (*a*) at the freezing point, (*b*) at the boiling point?

242. What is the largest diameter of pipe that will carry 200 g.p.m. of castor oil with laminar flow at 30°C.?

243. Two water reservoirs are connected by 2000 ft. of 12-in. pipe for which f is 0.038 and the flow produced by the difference in level is 6.0 c.f.s. If a new 12-in. pipe 1500 ft. long is laid from the higher reservoir, parallel to the old line and connected to the old line 1500 ft. from its inlet, compute the total quantity flowing. For the new pipe $f = 0.019$.

244. In Fig. 152 pipes *A*, *B* and *C* have diameters of 24, 18 and 12 in. and lengths of 2000, 3000 and 4000 ft., respectively. If the flow in pipe *A* is known to be 15 c.f.s., compute the head *H*. Use values of f for old cast-iron water pipe from Table VII.

[1] For discussion, photographs and bibliography on flow in bends, see W. Kaufman, "Hydromechanik," vol. II, p. 84, Julius Springer, Berlin, 1934.

245. A horizontal line of new cast-iron pipe, 4 in. in diameter and 600 ft. long, is connected to a pump which maintains a pressure of 50 lb. per sq. in. gage. What is the discharge of water (*a*) when the end of the pipe is open, (*b*) when it is fitted with a 1-in. nozzle in which the loss is 0.1 of the velocity head at exit? Use f from Table VII.

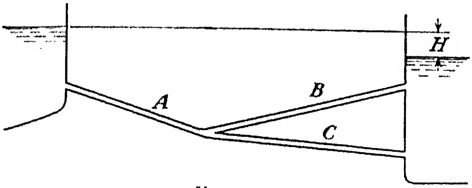


FIG. 152.

246. It is desired to pump 30 g.p.m. of oil through a pipe at a velocity which could be doubled and still have the flow remain laminar. The oil has a viscosity of 2.0 poises and its specific gravity is 0.8. Compute the diameter of the pipe.

247. An annular space between two tubes has a mean diameter of 3.75 in. and a thickness of 0.25 in. Assuming laminar flow, at what rate does water at 20°C. flow through this space with a pressure difference of 0.2 lb. per sq. in. in 100 ft.?

248. An oil is pumped through 2000 ft. of smooth 6-in. pipe at a velocity equal to twice the critical velocity. Compute the head required if $\mu = 0.6$ poises and $w = 58$ lb. per cu. ft.

249. A smooth 12-in. pipe 400 ft. long is connected to two reservoirs with a difference in level of 10 ft. The entrance and outlet of the pipe are square-

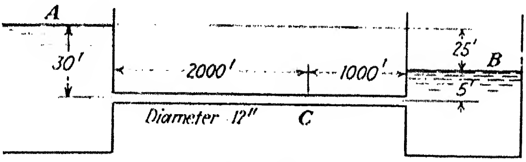


FIG. 153.

cornered and submerged. Compute the quantity of water at 20°C. which will flow through the pipe.

250. At a sudden enlargement of a water line from a diameter of 12 in. to one of 24 in., the hydraulic grade line rises 0.4 ft. Estimate the quantity flowing in the line.

251. In Fig. 153 the flow of water from reservoir A to reservoir B is 3.1 c.f.s. under a head of 25 ft. Is the pipe clean or old? If the line is broken wide open at C, what is the discharge at this point?

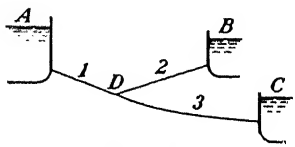


FIG. 154.

252. In Fig. 154 points A, B and D have elevations of 220, 205 and 150 ft., respectively. The pipes have lengths and diameters of $l_1 = 8000$ ft.,

$l_2 = 9000$ ft., $l_3 = 10{,}000$ ft., $d_1 = 12$ in., $d_2 = 18$ in. and $d_3 = 24$ in. The pressure at D is 15 lb. per sq. in. gage. Using values of f for new pipe from Table VII, find the elevation of the water surface at C.

253. An old 12-in. cast-iron pipe 1800 ft. long and a new 12-in. cast-iron pipe 2600 ft. long both connect a point at which the pressure is 10 lb. per sq. in. gage with a reservoir in which the water surface is 40 ft. higher. What is the diameter of an equivalent new pipe 2000 ft. long?

CHAPTER X

FLOW WITH A FREE SURFACE

117. Nature of Flow with a Free Surface.—The flow of liquid with a free surface is, in most cases, extremely complicated as compared with flow in pipes, the difficulties involved in the two subjects being comparable only in the very simplest case of uniform flow in channels, a condition requiring the cross section to be constant in area and form. Even after making the simplifying assumption of uniform flow, the wide range of forms of cross section and of conditions of channel surface makes it practically impossible to formulate a general description of the motion.

The free surface of a flowing fluid is under constant pressure since every part of the surface is exposed to the same atmosphere; the pressure being thus constant, the only force causing flow is the weight of the fluid or a component of it. The forces resisting motion are the viscosity forces when the flow is laminar or a combination of viscosity and inertia forces when the flow is turbulent.

With respect to the forces producing and resisting motion, the flow in pipes and uniform flow in channels with a smooth free surface are similar, but they are quite different in the manner in which the gravity forces are applied. In pipes flow is accompanied by a drop of pressure while uniform flow in channels is always accompanied by a change in elevation. When the flow in a channel is nonuniform it is not possible to treat the channel as a whole and it becomes necessary to account for difference in elevation, change in velocity and resistance to flow over every element of length of the channel.

Water is, of course, the usual fluid involved in flow with a free surface and the empirical coefficients and formulas offered in this chapter are based upon experiments with that liquid. Any theory presented is, however, applicable to any liquid. The use of open channels in transporting industrial liquids is not uncommon.

118. Hydraulic Slope.—The energy or head required to maintain flow in a channel is always obtained at the expense of potential energy. In uniform flow the velocity is constant, and, there being no change in kinetic energy, the entire change in elevation of a stream is chargeable to the maintaining of flow.

Figure 155*a* shows a longitudinal section of a channel in which the flow is uniform. Writing Bernoulli's equation for a stream tube connecting points 1 and 2 on the surface,

$$\frac{V_1^2}{2g} + h = \frac{V_2^2}{2g} + \text{(lost head)} \tag{1}$$

A similar equation for any stream tube lying parallel to the surface, such as that connecting points 3 and 4, is

$$\frac{V_3^2}{2g} + \frac{p_3}{w} + h = \frac{V_4^2}{2g} + \frac{p_4}{w} + \text{(lost head)} \tag{2}$$

By canceling equal terms in these equations it is seen that

$$h = \text{lost head} \tag{3}$$

from which it appears that the head lost is the same for all

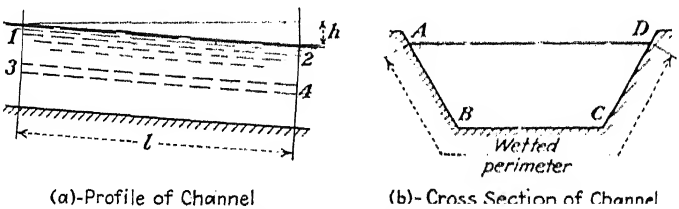


FIG. 155.—Flow in an open channel.

stream tubes and is equal to the drop in the free surface. The hydraulic slope of the free surface is the ratio

$$s = \frac{h}{l} \tag{4}$$

and, since h is the number of foot-pounds of potential energy given up by each pound of liquid in flowing a distance l, this ratio represents the head lost per unit length of channel. It is equal to the sine of the angle between the surface and a horizontal plane but in most cases it is practically equal to the tangent or to the angle itself.

In Fig. 155*a* the slope of the surface and the slope of the bottom are equal. Later in dealing with nonuniform flow it becomes

necessary to distinguish carefully between the slope of the channel bottom, the slope of the free surface and the ratio

$$s = \frac{\text{lost head}}{\text{length}}$$

119. Hydraulic Radius.—Resistance to flow is equal to the total shear force on the wetted surface. Channels having a shape in which the wetted surface is large in proportion to the cross section of the stream have a relatively large resistance, and vice versa; formulas involving resistance must therefore contain some expression of the relation between cross-sectional area and the area of the wetted walls of the channel. For this purpose the hydraulic radius, sometimes called the hydraulic mean depth, is introduced.

The hydraulic radius of a given cross section of channel is equal to the area of that cross section divided by its wetted perimeter. Thus in Fig. 155*b* the hydraulic radius is

$$R = \frac{\text{area } ABCD}{\text{length } ABCD}$$

The hydraulic radius might be called a section factor since it describes the shape of a section. It is not completely descriptive of a section and should be used with caution in dealing with unusual shapes of channels.

120. Open-channel Formulas.—Formulas dealing with flow in channels must contain terms representing the mean velocity V, the slope s, the proportions of the channel or the hydraulic radius R and the condition of the walls in contact with the liquid.

There is no entirely theoretical derivation for the relationship between these four variables and in developing formulas dealing with them it is necessary to assume the form of the equation and to insert three or more exponents or coefficients determined by experiments.

On the basis of experiments made in 1775, Chezy proposed a relationship that can be expressed in the form

$$V = C\sqrt{Rs} \tag{5}$$

in which C is a coefficient largely dependent upon the roughness of the walls. The term C is a dimensional coefficient, its dimension being $L^{1/2}/T$. Later experimenters, notably Darcy, Bazin,

and Ganguillet and Kutter, showed that Chezy's C also varies with R and developed empirical formulas for it containing R and also another quantity n, expressing the roughness. The formula of Ganguillet and Kutter, unfortunately and probably wrongly, also contained an s term. Its use became so general that the best known medium for stating the roughness of a channel is the so-called Kutter's n.

The most convenient formula adapted to the use of Kutter's n is that by Manning,

$$V = \frac{1.486}{n} R^{2/3} s^{1/2} \tag{6}$$

and by comparing this equation with Eq. (5) it appears that Manning's value for Chezy's C is

$$C = \frac{1.486 R^{1/6}}{n} \tag{7}$$

Because of its simplicity and its adaptation to Kutter's n, the Manning formula is being more widely used each year.[1]

A few values of Kutter's n are given below. These coefficients are based upon experiments with water at ordinary temperatures and they are not applicable to other liquids.

Type of Surface	Kutter's n
Planked surfaces	0.011
Smooth concrete surfaces	0.012
Smooth metal	0.011
Corrugated metal	0.022
Earth canals in good condition	0.025
Earth canals with stones or weeds	0.035

Open-channel formulas may be applied to pipes, in which case $R = d/4$. The Chezy formula and the Darcy formula for pipes, Eq. (20), page 202, when solved for V are similar in form.

121. Resistance to Flow.—The manner in which the walls of a channel resist flow of a liquid is the same as in the case of pipes. Here again the shear force at the walls sets up a complex motion within the fluid, the maintenance of which requires a continuous input of energy. When flow is uniform, the potential energy given up by the liquid as it flows downstream is just sufficient to

[1] For a discussion of open-channel formulas and tables of Kutter's n, see H. W. King, "Handbook of Hydraulics," McGraw-Hill Book Company, Inc., New York, 1929.

provide the required input of energy and there is no acceleration of the stream as a whole.

The average tractive force per unit area at the wall of the channel can be computed by considering the equilibrium of the body of liquid of length l between sections aa and bb, Fig. 156. The stream having a cross section A, the total weight of the body under consideration is wAl with a component parallel to the motion equal to $wAl \sin \theta = wAls$. The wetted perimeter being P, the area of contact is Pl and the total tractive force or resistance is $\tau Pl = F$, where τ is the tractive stress. The pressure forces on the two ends of the prism being equal and the condition being one of equilibrium, it follows that

$$F = \tau Pl = wAls \qquad (8)$$

and

$$\tau = w\frac{A}{P}s = wRs \qquad (9)$$

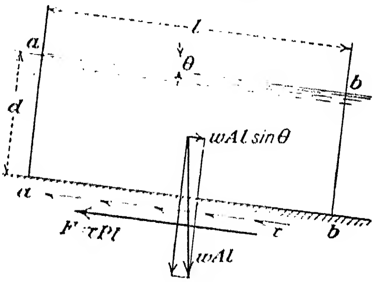


FIG. 156. Resistance to flow in a channel.

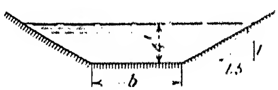
or, in words, the average shear force per unit area at the wall of the channel is the product of the specific weight of the liquid, the hydraulic radius of the channel and the slope. Substituting the empirical value of the product Rs as obtained from the Chezy and Manning formulas,

$$\tau = \frac{wV^2}{C^2} = \frac{wn^2}{(1.486)^2}\frac{V^2}{R^{1/3}} \qquad (10)$$

When a channel is wide in proportion to depth d and when the sides are short, the effect of the sides is small and R is approximately equal to the depth. The tractive force per unit area is then

$$\tau = wds \qquad (11)$$

Example. An earth canal of trapezoidal cross section having side slopes of 1.5 horizontal to 1 vertical is to be 4 ft. deep and is to carry 234 c.f.s. at a velocity of 4.5 ft. per sec. Assuming the canal to be in good condition, what is the required slope?



Solution. The cross section of the stream is $Q/V = 234 \div 4.5 = 52$ sq. ft. The area in terms of the bottom width b is

$$4b + (4 \times 6) = 52$$

whence $b = 7$ ft. The hydraulic radius is then

$$R = \frac{\text{area}}{\text{wetted perimeter}} = \frac{52}{7 + 2\sqrt{6^2 + 4^2}} = \frac{52}{21.42} = 2.43$$

Using the Manning formula, Eq. (6),

$$V = \frac{1.486}{n} R^{2/3} s^{1/2}$$

and $n = 0.025$ from page 236,

$$s = \frac{V^2 n^2}{(1.486)^2 R^{4/3}} = \frac{(4.5)^2 \times (0.025)^2}{2.208 \times (2.43)^{4/3}} = 0.00176$$

Problem 254. A V-shaped channel with side slopes of 1 to 1 has a maximum depth of 4 ft. when the discharge is 32 c.f.s. and the flow is uniform. Find the slope of the channel if it is lined with smooth concrete.

255. A canal is 20 ft. wide and has vertical sides. It carries 300 c.f.s. in uniform flow at a velocity of 5 ft. per sec. and it is lined with plank. What is the slope?

256. A semicircular flume of corrugated iron is 8 ft. in diameter and has a fall of 4 ft. in 2500 ft. of length. What is the discharge when it is flowing full? What is the average drag or shear force per square foot of surface?

257. An earth canal in good condition has side slopes of 1 to 1 and a depth equal to one-quarter of the bottom width. It carries 240 c.f.s. in uniform flow at a velocity of 3 ft. per sec. What is the slope?

258. A channel has a V-shaped cross section with side slopes of 1 to 1. When the depth is 4 ft. the discharge is 32 c.f.s. What is the discharge when the depth is 6 ft.? The flow is uniform in both cases.

122. Laminar Flow in Open Channels. The Chezy and Manning formulas of Art. 120 and Eq. (10) for tangential stress are applicable only to turbulent flow, in which case the slope or resistance is known to be nearly proportional to the velocity squared; these equations are not to be used in laminar flow.

There is abundant evidence that flow in channels is laminar at low velocities and it might be correctly supposed that there exists a critical velocity, in the sense in which the term is used in connection with flow in pipes,[1] below which the flow is laminar and above which it is turbulent. It is not possible to predict this limiting value for laminar flow in channels by comparison with Reynolds' criterion for pipes, as laminar flow is known to exist at much greater velocities in channels than in pipes of the same hydraulic radius.

[1] The term critical velocity will be used in an entirely different sense in the following articles dealing with flow in channels.

The loss or resistance in laminar flow is proportional to the velocity, that is, the slope is proportional to velocity, and the formula for velocity would be of the form

$$V = KR^2s \tag{12}$$

in which K might vary to some extent with R.

123. Velocity Distribution in Cross Section of a Channel.—The distribution of velocity in an open channel is affected by the traction on the walls in the same manner as in pipes and in addition it is influenced by the presence of the free surface. The velocity varies widely over any cross section, being greater at the point or points least affected by the solid boundaries and the free surface. Figure 157 shows a cross section of a channel and diagrams showing the distribution of velocity along two vertical lines *aa* and *bb* and a horizontal line *cc*.

The thread of highest velocity in the channel in this case is in the center of the section on line *aa* and at a depth of about one-fourth of d. The depth of the point of maximum velocity increases for relatively narrower channels and also increases for a position nearer to the side, such as line *bb* of Fig. 157. It has been shown conclusively that there is no slip of fluid along the walls of a pipe, the velocity at the wall being zero, and it is therefore certain that the velocity is also zero immediately at the walls of an open channel.

The distribution of velocity along the horizontal line *cc* is typical, being more nearly uniform at greater depths; the maximum is more prominent near the surface. At the surface the region of maximum velocity, commonly designated by the German term *thalweg*, is usually plainly visible.

It has been observed by Gibson[1] and others that the flow in a channel is spiral in nature, the rotational component of the velocity being as shown in Fig. 157, with downward flow at the center and upward flow at the side. The rotational velocity is small in comparison to the downstream component. Gibson also observed that the level of the surface is slightly higher at the *thalweg* than at the sides.

Even in the simpler forms of channels, for example the semicircular flume, the distribution is too complex to admit of mathe-

[1] GIBSON, A. H., "Hydraulics and Its Applications," 4th ed., p. 329, D. Van Nostrand Company, Inc., New York, 1930.

matical analysis. Irregularity of form or change of alignment of the channel or wind over the surface will alter the velocity distribution. At a bend in a channel there is an increase of velocity near the outside of the bend, the distribution of velocity

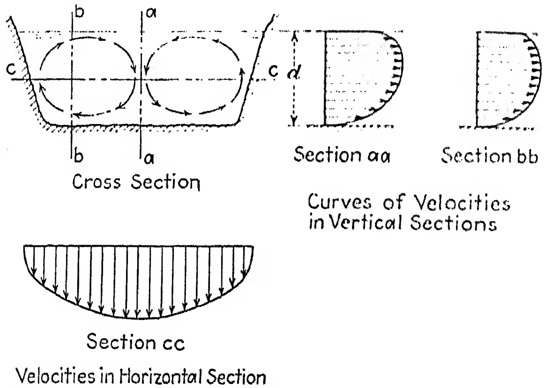


FIG. 157.—Velocity distribution in a straight channel.

along a horizontal section being as shown in Fig. 158, and the surface is slightly higher at the outside. Since the velocity is greater near the surface than near the bottom, the effect of centrifugal force is greater at the top. This establishes a spiral motion as indicated in Fig. 158 with an outward component of

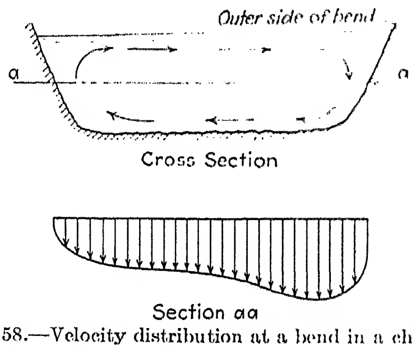


FIG. 158.—Velocity distribution at a bend in a channel.

velocity near the surface, a downward component at the outside and an inward component near the bottom.

The kinetic energy content of any stream of nonuniform velocity is greater than that computed by using the average velocity. The energy term in Bernoulli's equation is neverthe-

less written as $V^2/2g$ where V is the average velocity, thereby introducing an error which is usually, but not always, compensated for by the empirical coefficients. Assuming the average kinetic energy per pound of flowing liquid to be $V^2/2g$, the total kinetic energy per second is $QwV^2/2g$. It will be recalled from Art. 101 that the corrected total kinetic energy per second in pipes is $2QwV^2/2g$ for laminar flow and about $1.1QwV^2/2g$ for turbulent flow. It is likely that equal or larger corrections should apply to flow in channels.

124. Specific Energy and Critical Depth.—The total energy per unit weight of liquid flowing in a channel, referred to the bottom of the channel as datum, is known as the specific energy. In Fig. 159 consider any particle at a depth d_1 below the free surface and having a velocity V_1. The total energy per unit weight, referred to the bottom immediately below point 1, is found from the Bernoulli constant to be

$$E = \frac{V_1^2}{2g} + \frac{p_1}{w} + (y - d_1) \qquad (13)$$

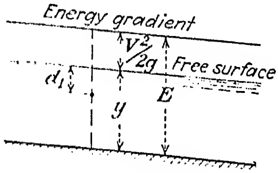


FIG. 159.—Specific energy.

Assuming the pressure to be the same as for static conditions, $p_1 = wd_1$ and this equation becomes

$$E = \frac{V_1^2}{2g} + y \qquad (14)$$

Replacing V_1 by the average velocity V, the value of E for any particle is

$$E = y + \frac{V^2}{2g} \qquad (15)$$

This quantity, under the assumption of uniform distribution of velocity in a cross section, is a constant for all particles in a cross section and in uniform flow is the same at all sections of the channel. It is the specific energy. The quantity has the dimension of a length and, when the distance E is plotted from section to section, the upper line in Fig. 159, marked "Energy Gradient," is produced.

Substituting for V the expression Q/A, Eq. (15) becomes

$$E = y + \frac{Q^2}{2gA^2} \qquad (16)$$

For a unit width of rectangular channel with a constant discharge per unit width equal to q, the velocity is $V = q/y$ and the specific energy may be written

$$E = y + \frac{q^2}{2gy^2} \tag{17}$$

Each term of Eqs. (15), (16) and (17) is represented in Fig. 160, in which the specific energy E is plotted as abscissas against ordinates representing depth. At very small values of y the curve

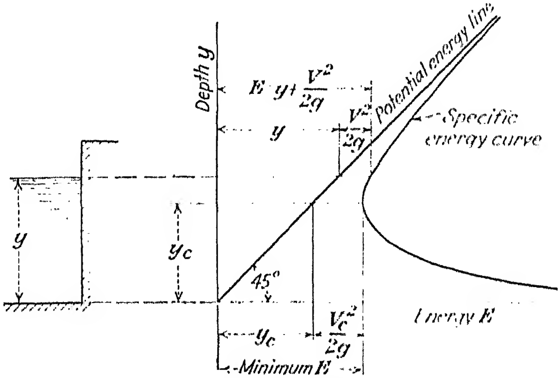


FIG. 160.—Specific energy diagram.

is asymptotic to the horizontal axis and at large values of y it approaches a 45-deg. line, the equation of which is

$$E_p(\text{potential energy}) = y$$

That part of any abscissa between the potential energy line (45-deg. line) and the curve represents the kinetic energy term, $V^2/2g$, $Q^2/2gA^2$ or $q^2/2gy^2$. For either large depths or very small depths, the latter requiring excessive velocity, the specific energy becomes very large.

An examination of the curve of Fig. 160 shows that there are two depths for a given value of the specific energy and a single depth at which the energy is a minimum. A very special interest attaches to the depth for minimum specific energy. This minimum can be found by the usual procedure of placing the first derivative equal to zero, which will now be done for the simple case of the rectangular channel. The algebra becomes very

involved for the trapezoidal channel. The first derivative of E with respect to y is

$$\frac{dE}{dy} = -\frac{q^2}{gy^3} + 1 = 0 \tag{18}$$

whence

$$y = \sqrt[3]{\frac{q^2}{g}} \tag{19}$$

and, since $q = Vy$, the velocity for minimum specific energy is

$$V = \sqrt{gy} \tag{20}$$

From Eq. (20), $y = V^2/g$ and, substituting this in Eq. (15),

$$E = \frac{V^2}{2g} + \frac{V^2}{g}$$

from which it may be seen that, for minimum specific energy in a rectangular channel,

$$\text{Potential energy} = y = \frac{2}{3}E \tag{21}$$

and

$$\text{Kinetic energy} = \frac{V^2}{2g} = \frac{1}{3}E \tag{22}$$

The depth at which this condition of minimum specific energy obtains is called the critical depth y_c and the corresponding velocity is the critical velocity.

Up to this point this article has dealt largely with the specific energy content for a given quantity q. The equations will now be examined with the object of determining the q for a given specific energy. Equation (17) may be written in the form

$$q^2 = 2g(Ey^2 - y^3) \tag{23}$$

This equation being a cubic in y, it might appear that there are three depths at which a given q can flow with a given E and there will be two maximums or minimums of q for a given E. Taking the first derivative of q with respect to y and equating it to zero,

$$\frac{dq}{dy} = \frac{g}{q}(2Ey - 3y^2) = 0 \tag{24}$$

whence

$$2Ey - 3y^2 = 0 \tag{25}$$

and

$$y = 0 \qquad \text{and} \qquad y = \tfrac{2}{3}E \tag{26}$$

An examination of these roots shows that q is a maximum when $y = \frac{2}{3}E$ and confirms the result that might well have been expected, namely, that the conditions for maximum q with a given E are precisely the same as for a minimum E with a given q, and both are found at critical depth.

Solving Eq. (23), it is found to be satisfied by two positive real values of y and one negative value of no physical significance. Of the two positive real roots, one is always greater than y_c and the other is less than y_c. The velocity at critical depth is

$$V_c = \sqrt{gy_c} \tag{27}$$

and it follows from an examination of Eq. (15) and Fig. 160 that at depths greater than critical the velocity is such that $V < \sqrt{gy}$ and for depths less than critical $V > \sqrt{gy}$.

When the depth is greater than critical the flow is said to be tranquil or streaming; when the depth is less than critical the velocity is high and the motion is described as shooting or rapid flow.

It is well to note that although the specific energies may be equal for two corresponding depths, one with tranquil flow and the other with shooting flow, the energy input required to maintain flow is quite different in the two cases, a much greater slope being required for continued shooting flow.

Example.—A rectangular channel lined with concrete and 20 ft. wide carries 400 c.f.s. at a velocity of 5 ft. per sec. What is the specific energy? At what depth would this quantity flow with minimum specific energy?

Solution.—The area of the channel is $Q/V = 400 \div 5 = 80$ sq. ft. Then the depth y is $80 \div 20 = 4$ ft. The specific energy is

$$E = \frac{V^2}{2g} + y = \frac{5^2}{64.4} + 4 = 0.39 + 4 = 4.39 \text{ ft.}$$

The velocity of 5 ft. per sec. is less than $\sqrt{gy}$ so the flow is streaming, that is, the depth is greater than critical.

The specific energy will be a minimum when $V = V_c = \sqrt{gy_c}$. The depth and velocity are then critical and

$$\frac{V_c^2}{2g} = \frac{1}{3}E = \frac{1}{3}\left(\frac{V_c^2}{2g} + y_c\right)$$

But $V_c = Q/A = 400/20y_c = 20/y_c$ and, substituting this for V_c,

$$\frac{20^2}{64.4y_c^2} = \frac{1}{3}\left(\frac{20^2}{64.4y_c^2} + y_c\right) \qquad \text{and} \qquad y_c = 2.32 \text{ ft.}$$

Then $V_c = 20 \div 2.32 = 8.62$ ft. per sec. and the minimum specific energy is

$$\frac{V_c^2}{2g} + y_c = 1.16 + 2.32 = 3.48 \text{ ft.}$$

Problem 259. A rectangular channel 20 ft. wide and 4 ft. deep carries 400 c.f.s. What is the specific energy? Is the depth more or less than critical depth?

260. In a rectangular channel 20 ft. wide what is the maximum flow for a specific energy of 6 ft.? What slope is necessary if the channel is lined with concrete?

261. At what two depths will a rectangular channel 30 ft. wide carry 360 c.f.s. with a specific energy of 4.5 ft.?

125. Nonuniform Flow. The foregoing articles of this chapter have dealt entirely with uniform flow, a condition requiring a cross section constant in both area and form and hence a surface slope parallel to the bottom slope. In a particular form of channel and for a given quantity and slope, there is only one depth at which flow will be uniform. This is called the normal depth. There is, however, an unlimited number of ways in which the same quantity of water might be made to flow through the same channel with a variable depth and a surface slope different from the slope of the bottom. If the depth is controlled so as to be greater than normal, the effective slope is less than the bottom slope, and conversely depths less than normal require effective slopes greater than the bottom slope. In this connection the term effective slope is used to designate the slope of the energy gradient, line *mn* of Fig. 161.

If the depth is other than normal, the surface profile is a curve, the equation of which can be developed by making the usual assumption of uniform velocity in a cross section and computing the loss of energy in the same manner as for uniform flow, that is, by the Chezy or some other channel formula.

Figure 161 illustrates a case in which the velocity is being reduced, the bottom having a slope i and the energy gradient a slope s. The total head at any section referred to the horizontal datum is the ordinate to the energy gradient

$$E = K + y + \frac{V^2}{2g} \tag{28}$$

The total head at a section downstream a distance dx is

$$E' = (K - i\,dx) + (y + dy) + \frac{V^2}{2g} + d\left(\frac{V^2}{2g}\right) \tag{29}$$

in which $i\,dx$ is an increment of fall, dy is an increment of depth and $d(V^2/2g)$ is a decrement of velocity head. In the figure dh is an increment of lost head and the change in total head in distance dx is $E - E' = dh$. Referring to Arts. 118 and 120, it will be noted that the head lost per foot of distance traveled is the slope of the energy gradient, in this case dh/dx. Assuming now

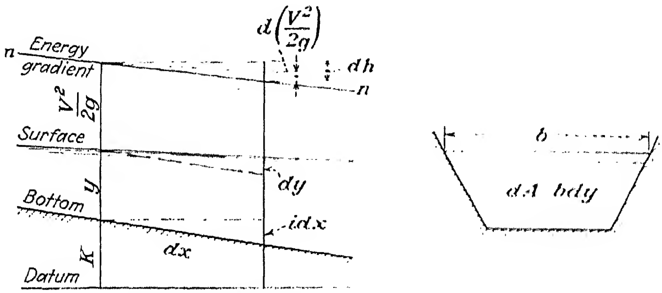


FIG. 161.—Equation of nonuniform flow.

that loss in nonuniform flow is the same as for uniform flow and using the Chezy formula, $V = C\sqrt{Rs}$

$$\frac{V^2}{C^2R} = \frac{dh}{dx} \tag{30}$$

From the geometry of Fig. 161, or by letting $dh = E - E'$, and subtracting Eq. (29) from Eq. (28), it follows that

$$dh = i\,dx - dy \tag{31}$$

and

$$s = \frac{V^2}{C^2R} = \frac{dh}{dx} = i - \frac{dy}{dx} - \frac{d}{dx}\left(\frac{V^2}{2g}\right) \tag{32}$$

This general equation for channel flow can be simplified by introducing $V = Q/A$ and noting that the increment of area is $dA = b\,dy$, in which b is the surface width of the channel. The last term of Eq. (32) then becomes

$$\frac{d}{dx}\left(\frac{V^2}{2g}\right) = \frac{d}{dx}\left(\frac{Q^2}{2gA^2}\right) = \frac{-2Q^2}{2gA^3}\frac{dA}{dx} = \frac{-Q^2b}{gA^3}\frac{dy}{dx} \tag{33}$$

Substituting this expression in Eq. (32),

$$s = i - \frac{dy}{dx} + \frac{Q^2b}{gA^3}\frac{dy}{dx} \tag{34}$$

and

$$\frac{dy}{dx} = \frac{i - s}{1 - \dfrac{Q^2b}{gA^3}} = \frac{i - \dfrac{Q^2}{A^2C^2R}}{1 - \dfrac{Q^2b}{gA^3}} \tag{35}$$

The expression dy/dx is the convergence or divergence of the free surface from the plane of the channel bottom. Although it rarely is possible to integrate this equation, it is not without value, because an examination of it will often reveal helpful facts about the surface profile. For this purpose it may be simplified by substituting the velocity $V = Q/A$ and introducing the mean depth $D = A/b$. It is then written

$$\frac{dy}{dx} = \frac{i - s}{1 - \dfrac{V^2}{gD}} = \frac{i - \dfrac{V^2}{C^2R}}{1 - \dfrac{V^2}{gD}} \tag{36}$$

On inspecting this equation it appears that

1. When i is greater than the slope s necessary for uniform flow at the given depth, the numerator is positive in sign, and vice versa.

2. When $V > \sqrt{gD}$ the denominator is negative and when $V < \sqrt{gD}$ the denominator is positive.

3. At critical depth, that is, when $V = \sqrt{gD}$, the denominator is zero and dy/dx is mathematically infinite.

It appears then that Eq. (36) may represent a great number of surface profile forms. The third statement is of special interest in explaining the fact that water flowing at critical depth has an unstable wavy surface. The depth can never be exactly critical over any finite length of channel. At this stage any small variations of D, such as may be caused by minor irregularities which are always present, will result in values of dy/dx that are alter-

nately negative and positive. The surface is then wavy and unstable.

The surface in nonuniform flow may assume a variety of forms, depending upon the way in which the flow is controlled by inlet and outlet structures, dams or other obstructions and changes in grade of the canal. These profiles may be determined by applying Eq. (35) to short reaches of the channel or by applying the Bernoulli equation from point to point, for example, to sections 1 and 2 of the channel illustrated in Fig. 162. Let l be any convenient length, keeping it small enough so that there is not a wide difference between conditions at the two sections, and let V_m and R_m be the mean velocity and mean hydraulic radius for

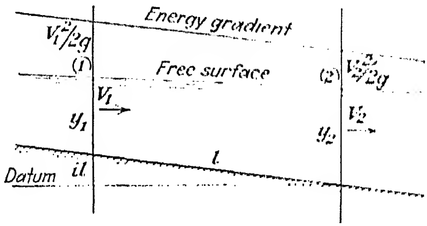


FIG. 162.—Nonuniform flow.

the reach l. The head lost between sections as computed by the Chezy formula, $V = C\sqrt{Rs}$, is then

$$h = sl = \frac{V_m{}^2 l}{C^2 R_m}$$

and the Bernoulli equation corrected for loss is

$$\frac{V_1^2}{2g} + (il + y_1) = \frac{V_2^2}{2g} + y_2 + \frac{V_m{}^2 l}{C^2 R_m} \tag{37}$$

In obtaining any quantity from this equation a solution by trial is necessary. The value of V_m is computed either as the mean of V_1 and V_2 or as Q/A where A is the area of a cross section having a depth equal to the mean of y_1 and y_2. Likewise R_m either may be taken as a mean or may be based on a mean section. Neither method of computing V_m and R_m is strictly correct but the results are acceptable if the reach l is short.

Problem 262. A smooth concrete-lined channel has side slopes of 1.5 horizontal to 1 vertical and a bottom width of six times the depth. If $Q = 400$ c.f.s. and $s = 0.0001$, what is the normal depth?

263. A canal is regulated so that the slope of the energy gradient is 0.0008. The bottom slope is 0.006 and at a point where the depth is 6 ft. the velocity is 9 ft. per sec. At what rate is the depth changing? Is the stream deeper or less deep downstream?

126. Hydraulic Jump.—Under favorable circumstances flow at a depth less than critical depth may suddenly change to flow at a certain depth greater than critical with a corresponding reduction in velocity. This abrupt change in depth is known as the

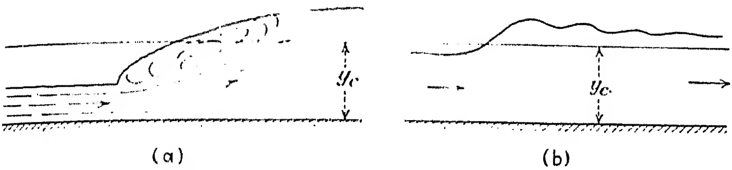


Fig. 163. Types of hydraulic jump.

hydraulic jump. The phenomenon consists of a sudden vertical expansion of the stream in the form shown in Fig. 163*a* if the jump is high, or of the form shown in Fig. 163*b* if the initial depth is not much below critical. In the first case the liquid is rolling back down the face of the jump and this eddying portion contains much entrained air. If colored liquid is added to this face, several seconds are required for it to become clear again. In the second case the surface is waving but not broken and there is no back roll.

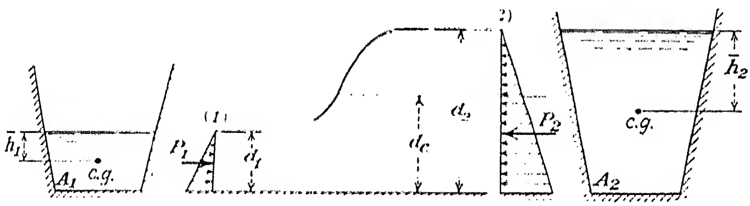


Fig. 164. Hydraulic jump theory.

The relation between the depths and velocities upstream and downstream from the jump can be developed by applying the principles of impulse and momentum. Figure 164 represents a profile taken through a hydraulic jump in a horizontal channel. The only forces capable of causing a change in momentum are the pressure forces exerted by the water on the left of section 1 and on the right of section 2. Assuming these pressure forces to be the same as for static conditions, they are respectively

$P_1 = w\bar{h}_1A_1$ and $P_2 = w\bar{h}_2A_2$, in which $\bar{h}$ is the depth of the center of gravity of the cross-sectional area A. The momentum of the quantity Q which passes section 1 in 1 sec. is QwV_1/g and the momentum of the same Q as it leaves through section 2 is QwV_2/g. Neglecting tangential forces at the walls of the channel, the only force in the direction of motion is $P_1 - P_2$. By equating this to the change of momentum per second,

$$w\bar{h}_1A_1 - w\bar{h}_2A_2 = \frac{QwV_2}{g} - \frac{QwV_1}{g} \tag{38}$$

Substituting velocity in terms of Q and A, canceling w and transposing terms, this equation becomes

$$\frac{Q^2}{gA_1} + \bar{h}_1A_1 = \frac{Q^2}{gA_2} + \bar{h}_2A_2 \tag{39}$$

Considering only a unit width of a rectangular channel, the pressure forces are $wd_1^2/2$ and $wd_2^2/2$; the momentum is $wV_1^2d_1/g$ before the jump and $wV_2^2d_2/g$ after. The impulse and momentum equation corresponding to Eq. (38) is then

$$\frac{wd_1^2}{2} - \frac{wd_2^2}{2} = \frac{wV_2^2d_2}{g} - \frac{wV_1^2d_1}{g} \tag{40}$$

which can be written

$$\frac{d_1V_1^2}{g} + \frac{d_1^2}{2} = \frac{d_2V_2^2}{g} + \frac{d_2^2}{2} \tag{41}$$

Using the continuity equation $V_1d_1 = V_2d_2$ and after considerable manipulation of Eq. (41), the expressions for d_1 and d_2 are found to be

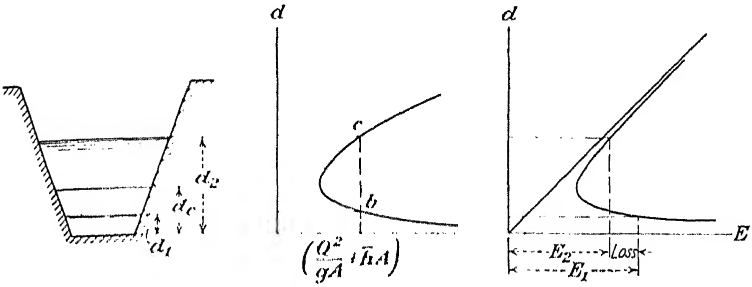
$$d_1 = -\frac{d_2}{2} + \sqrt{\frac{2d_2V_2^2}{g} + \frac{d_2^2}{4}} \tag{42}$$

and

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{2d_1V_1^2}{g} + \frac{d_1^2}{4}} \tag{43}$$

If $V_1 = \sqrt{gd_1}$, the condition for critical depth, is substituted in Eq. (43), it is found that $d_2 = d_1$, that is, there is no jump. It can be proved from Eqs. (42) and (43) that the jump can take place only from an initial depth less than critical depth.

The solution of Eq. (39) to find the height of jump in a channel not rectangular is far more complicated than the above expressions for d_1 and d_2. The solution for any given form of channel may be expedited by treating the expression $\frac{Q^2}{gA} + \bar{h}A$, from Eq. (39), as a function of the depth, which function, for lack of a better name, may be called the momentum function since it is obtained from the momentum equation, Eq. (38). Since the factor w was canceled, the momentum function does not have the dimension of momentum. Bakhmeteff[1] plots this momentum



(a)-Channel Section (b)-Momentum Function (c)-Specific Energy
FIG. 165. Solution of hydraulic jump equations.

function against depth in a diagram of the form shown in Fig. 165. The curve is drawn by plotting values of $\frac{Q^2}{gA} + \bar{h}A$ computed for any given Q and various depths. The depth d_2 to which the flow may change from any initial depth d_1 is found by drawing a vertical line bc through the point on the curve where $d = d_1$, the upper intersection with the curve being at depth d_2. The depths d_1 and d_2 are not depths of equal energy since the jump is accompanied by a considerable loss of energy, which may be computed by writing the Bernoulli equation between points 1 and 2. The loss could also be scaled from the specific energy diagram, Fig. 165c, where it is represented by the difference between the specific energy before and after the jump, $E_1 - E_2$.

127. Examples of Critical Depth. In Art. 124 it was explained that flow at critical depth is flow having a minimum specific

[1] BAKHMETEFF, BORIS A., "Hydraulics of Open Channels," McGraw-Hill Book Company, Inc., New York, 1932. This book contains a very complete discussion of the hydraulic jump.

energy content for a given Q or a maximum Q for a given specific energy. An example of critical depth may be found at the entrance to a steep channel. The flow from a reservoir into a channel is controlled by the critical depth at entrance provided the slope of the bed is such that the channel is capable of taking away all incoming flow at a depth less than critical. Under these conditions, illustrated in Fig. 166, the available head D can be safely assumed to be divided naturally into velocity head and depth in that proportion which will produce maximum flow.

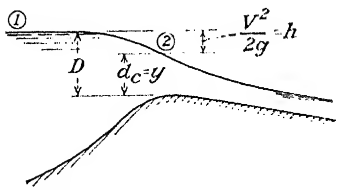


FIG. 166.—Critical depth at entrance.

Writing the Bernoulli equation for points 1 and 2, while neglecting velocity head at 1 and the loss between 1 and 2,

$$D = \frac{V^2}{2g} + y \tag{44}$$

Letting $V^2/2g = h$, the velocity is $V = \sqrt{2gh}$. The area per unit width of channel at section 1 is $D - h$, making the quantity per unit width equal to

$$q = \sqrt{2gh}(D - h) = \sqrt{2g}(Dh^{\frac{1}{2}} - h^{\frac{3}{2}}) \tag{45}$$

To find the value of h which will produce maximum q, the first derivative of q with respect to h is equated to zero. Thus

$$\frac{dq}{dh} = \sqrt{2g}\left(\frac{1}{2}Dh^{-\frac{1}{2}} - \frac{3}{2}h^{\frac{1}{2}}\right) = 0 \tag{46}$$

and

$$h = \frac{D}{3}$$

Then

$$y = \frac{2}{3}D = 2h = 2\frac{V^2}{2g}$$

and

$$V = \sqrt{gy} \tag{47}$$

Equation (47) expresses the condition for flow at critical velocity, which now appears to exist at the entrance to a channel of ample slope. Then y is the critical depth d_c.

Figure 167 shows the profile of a channel with a large increase in slope. The upstream part has a slope for which the normal depth

is greater than d_c while the normal depth for the steeper downstream portion is less than d_c. The depth at some point slightly upstream from the break in grade must be equal to d_c.

The free overfall illustrated in Fig. 168 is an extreme case of the change in bottom slope shown in Fig. 167 and the depth will be critical some distance upstream from the drop. Critical-depth relations have been developed in this chapter on the assumption of a static distribution of pressure. The pressure downstream from the critical-depth section in Fig. 168 is progressively less than static and is zero in the freely falling sheet. The thickness t of the overfalling sheet is not far different from the end depth d_e, and the elevation of the center of the stream at overfall is approximately $t/2$. Assuming the velocity to be uniform in the

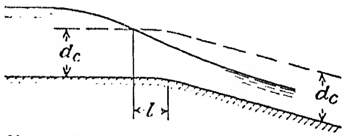


Fig. 167. Critical depth at change of slope.

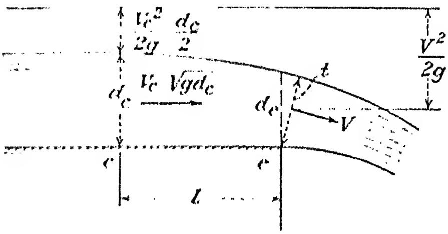


Fig. 168. Free discharge of channel.

falling sheet and neglecting the small loss of head, Bernoulli's equation between the centers of sections c and e is

$$\frac{V_c^2}{2g} + \frac{d_c}{2} + \frac{d_c}{2} = \frac{V^2}{2g} + \frac{t}{2} \tag{48}$$

From the continuity equation and the relation $V_c = \sqrt{gd_c}$ the discharge per unit width is

$$q = d_cV_c = d_c\sqrt{gd_c} = Vt \qquad \text{and} \qquad V = \frac{\sqrt{gd_c^3}}{t}$$

Substituting $V^2/2g = d_c^3/2t^2$ and $V_c^2/2g = d_c/2$ in Eq. (48), it reduces to

$$t^3 - 3d_ct^2 + d_c^3 = 0 \tag{49}$$

This equation is satisfied by the root $t = 0.65d_c$, which agrees very well with the value $d_e = \frac{2}{3}d_c$ since d_e is necessarily slightly

larger than t. The latter value was obtained by O'Brien,[1] using momentum theory, which might be expected to agree well with the energy theory employed here where the lost energy is small. From a series of experiments on water with a free overfall O'Brien found that

$$d_e = 0.643d_c \tag{50}$$

and also found the discharge to be given by the empirical formula

$$q = 11.0d_e^{1.61} \tag{51}$$

From the theoretical considerations of this article q might be estimated at

$$q = \sqrt{gd_c^3} = \sqrt{\frac{gt^3}{(0.65)^3}} = 10.83t^{1.5}$$

In the same experiments upon which Eqs. (50) and (51) are based it was determined that d_c is upstream from the fall a distance $l = 11.6d_c$.

128. Weirs.—The weir is a device widely used for measuring or controlling the flow of water in channels. The term is applied to overflow structures and devices of many shapes and arrangements, a few of which are shown in Fig. 169. Such weirs as those shown in Fig. 169*c* and *d* are overflow sections of dams intended primarily to control the depth of water upstream from the dam while those illustrated in Fig. 169*a* and *e* are usually used in measuring flow. The latter are sharp-crested weirs, the crest being of metal with a trim right-angled edge.

Flow over weirs is characterized by a drawing down or contraction of the free surface immediately upstream from the weir where the velocity is increasing rapidly. If the upper edge or crest of the weir is sharp or of a very small radius, the liquid breaks away from it and there is a crest contraction (line *ed*, Fig. 169*a*). If the ends, as in Fig. 169*b*, are of the same sharp-edged form, there is also an end contraction. The dimensions of the overfalling sheet or nappe are reduced by these contractions. The entire body of water upstream from the weir is moving and a large part of it must change the direction of its motion in approaching the opening. This change does not take place suddenly because acceleration

[1] O'BRIEN, M. P., Analyzing Hydraulic Models for Effect of Distortion, *Eng. News-Record*, Sept. 15, 1932, p. 313.

cannot be infinite; on the contrary, the change is gradual, the path of a particle is curved, and the nappe is contracted where the liquid rounds the crest.

The horizontal dimension L is called the length of the weir or nappe; the vertical distance H from the crest to the free surface,

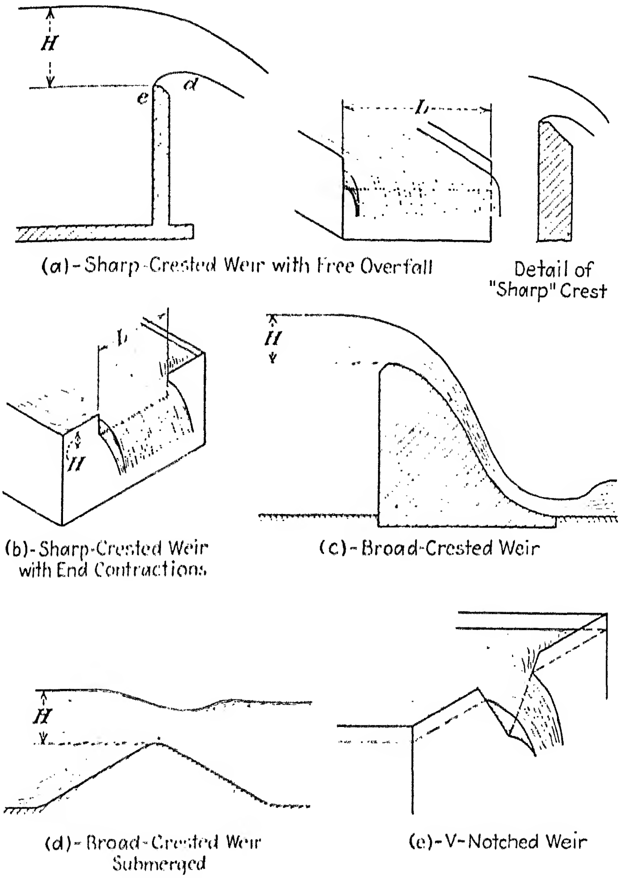


FIG. 169. Types of weirs.

measured at a point far enough upstream to be unaffected by the surface contraction, is called the head. The average velocity in the cross section of the channel at which H is measured is termed the velocity of approach. Computations of discharge over a weir are usually based upon measured values of H, L and the cross

section of the channel, but are always dependent upon certain empirical data obtained by calibration of the weir in question or similar ones.

129. Sharp-crested Rectangular Weir.—The discharge Q over a weir is dependent principally on the length L of the crest and the measured head H. In developing the relation between Q, H and L it is necessary to consider the kinetic energy of the water as it approaches the weir. In Fig. 170, section bb is taken to be upstream so that the depth is only slightly affected by the contraction of the surface. The average velocity V at section bb is called the velocity of approach.

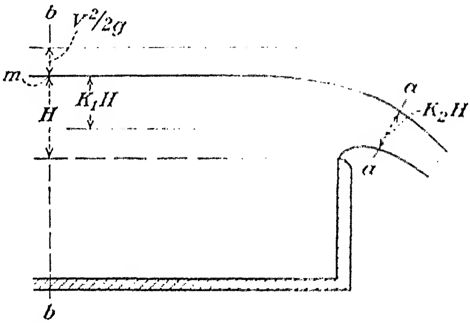


FIG. 170.—Sharp-crested rectangular weir.

Section aa is the section where the nappe is no longer affected by surface contraction or crest contraction. The nappe continues to contract below section aa, however, because its velocity increases as it falls. Beyond aa the nappe has acceleration g and it is surrounded by air at atmospheric pressure. The pressure in this part of the nappe is therefore taken to be atmospheric. The average velocity at aa is designated by U.

It has been shown experimentally that the coordinates of the surface profile immediately upstream, measured from the crest, and the dimensions of the nappe downstream from the weir are proportional to H. The thickness of the nappe measured along aa can therefore be expressed as K_2H where K_2 is some constant. The quantity K_1H is the distance from the free surface to the center of the mass passing through aa in any unit of time, the location of this center depending partly upon the distribution of velocity in section aa.

The values of kinetic energy at bb and aa are $V^2/2g$ and $U^2/2g$ ft. lb. per lb., respectively. Since these values are based on average velocities, their use introduces some error.

Bernoulli's equation between this point in aa and any point in section bb, for example, m, is

$$\frac{V^2}{2g} + K_1H = \frac{U^2}{2g} \tag{52}$$

from which the velocity in the nappe at aa is

$$U = \sqrt{2g}\sqrt{K_1H + \frac{V^2}{2g}} = \sqrt{2gK_1H}\sqrt{1 + \frac{V^2}{2gK_1H}} \tag{53}$$

If the velocity of approach is neglected, the velocity at aa is

$$U_1 = \sqrt{2gK_1H} \tag{54}$$

The cross section of the nappe at aa is K_2HL and from the continuity equation the total quantity per second is

$$Q = K_2HLU = K_2HL\sqrt{2gK_1H}\sqrt{1 + \frac{V^2}{2gK_1H}} \tag{55}$$

and

$$Q = K_2\sqrt{2gK_1}LH^{\frac{3}{2}}\sqrt{1 + \frac{V^2}{U_1^2}} \tag{56}$$

The terms V^2 and U_1^2 were introduced with the assumption that the velocity is uniform through both sections bb and aa. This is only approximated in the nappe and does not agree with the facts in section bb; therefore it is necessary to modify the ratio V^2/U_1^2 by a coefficient α, making it read $\alpha V^2/U_1^2$. Letting A be the cross section of the channel of approach, the velocity of approach can be written

$$V = \frac{Q}{A} = U_1\frac{K_2HL}{A}$$

whence

$$\frac{\alpha V^2}{U_1^2} = \alpha K_2^2\left(\frac{LH}{A}\right)^2 \tag{57}$$

Placing this in Eq. (56) and modifying the whole by a coefficient C' to correct for lost head, the expression for discharge is

$$Q = C'K_2\sqrt{2gK_1}\,LH^{3/2}\sqrt{1 + \alpha K_2^2\left(\frac{LH}{A}\right)^2} \tag{58}$$

Letting $C'K_2\sqrt{2gK_1} = C$, expanding the radical quantity and retaining only the first two terms of the expansion because the sum of the following terms is very small, this becomes

$$Q = CLH^{3/2}\left[1 + \frac{\alpha K_2^2}{2}\left(\frac{LH}{A}\right)^2\right] \tag{59}$$

The four coefficients C', K_1, K_2 and α in these equations have not been determined separately, but empirical values of the products $C = C'K_2\sqrt{2gK_1}$ and $\alpha K_2^2/2$ appear in the formula developed by King,[1] which is

$$Q = 3.34LH^{1.47}\left[1 + 0.56\left(\frac{LH}{A}\right)^2\right] \tag{60}$$

This formula is based chiefly on experiments by Francis[2] and Bazin,[3] giving more weight to the extensive work by Bazin who himself proposed the formula

$$Q = LH^{3/2}\left(3.248 + \frac{0.079}{H}\right)\left[1 + 0.55\left(\frac{LH}{A}\right)^2\right] \tag{61}$$

The Francis formula as proposed by him is

$$Q = 3.33L\left[\left(H + \frac{V^2}{2g}\right)^{3/2} - \left(\frac{V^2}{2g}\right)^{3/2}\right] \tag{62}$$

When reduced to the form of Eq. (59) the Francis formula becomes approximately

$$Q = 3.33LH^{3/2}\left[1 + 0.26\left(\frac{LH}{A}\right)^2\right] \tag{63}$$

There are numerous other empirical weir formulas[4] which cannot be given here.

[1] KING, H. W., "Handbook of Hydraulics," McGraw-Hill Book Company, Inc., New York, 1929.

[2] FRANCIS, J. B., "Lowell Hydraulic Experiments," D. Van Nostrand Company, Inc., New York, 1871.

[3] BAZIN, H., *Ann. ponts chaussées*, 1888.

[4] For a summary of weir formulas and experimental work, see footnote 1, above, and W. Kauffman, "Hydromechanik," vol. II, Julius Springer, Berlin, 1934.

The quantities in these formulas and those in the following pages are in foot and second dimensions. The constants are determined by experiments on water and are not necessarily applicable to other liquids.

Example.—A rectangular channel 30 ft. wide is to be regulated by a sharp-crested weir so that the water upstream from the weir is 4.5 ft. deep when $Q = 125$ c.f.s. The weir extends entirely across the channel and the overfall is free. How high is the weir?

Solution.—Using the King formula, Eq. (60),

$$Q = 3.34LH^{1.47}\left[1 + 0.56\left(\frac{LH}{A}\right)^2\right]$$

and, substituting the given data,

$$125 = 3.34 \times 30H^{1.47}\left[1 + 0.56\left(\frac{30H}{30 \times 4.5}\right)^2\right]$$

The last equation is rather difficult to solve algebraically. The factor in brackets however is only slightly more than unity. If it is taken as unity temporarily, then

$$125 = 3.34 \times 30H^{1.47} \qquad \text{and} \qquad H = 1.163$$

approximately. This value of H is a little too large but may be used, slightly reduced if desired, to evaluate the bracketed quantity. Thus

$$125 = 3.34 \times 30H^{1.47}\left[1 + 0.56\left(\frac{30 \times 1.16}{30 \times 4.5}\right)^2\right]$$

The quantity in the brackets is 1.037 and, solving for H,

$$H^{1.47} = \frac{125}{3.34 \times 30 \times 1.037} = 1.203$$

and

$$H = 1.134$$

A repetition of the process will give a new H more closely representing the formula but probably not more accurate.

130. End Contractions. A weir with end contractions is illustrated in Fig. 171. The edges at the ends are sharp like the crest. The nappe is contracted at the ends as well as at the crest and at the surface, and the cross section of the nappe is reduced in its horizontal dimension by the amount of the contractions. This contraction was found by Francis to be about $0.1H$ at each end, making the effective length of the weir $0.2H$ less than the actual

length of crest. Then the value of L which must be used in the equations of Art. 129 is

$$L = L' - 0.2H \tag{64}$$

This correction is not entirely satisfactory and end contractions are to be avoided where the weir cannot be calibrated. It is unsafe to use ordinary weir formulas with this correction if $L < 3H$. To insure complete contraction the end of the weir should be at a distance of at least $2H$ from the sides of the channel. F. Frese and the Swiss Society of Architects and Engineers offer rather cumbersome formulas which apply to contracted weirs of all proportions.[1] When the construction is such as to eliminate or suppress end contractions, the weir is said to be a suppressed weir.

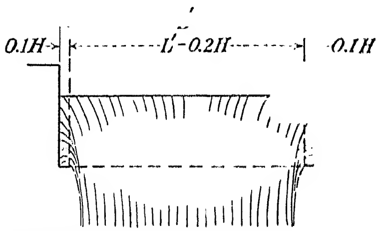


FIG. 171.—Correction for end contraction.

Problem 264. A sharp-crested weir 2.5 ft. high extends across the end of a channel 20 ft. wide with vertical sides. The water upstream from the weir is 3.75 ft. deep. Compare the discharges computed by formulas (60), (61) and (63).

265. A sharp-crested weir 2.5 ft. high with a crest 10 ft. long is installed at the end of a rectangular channel 20 ft. wide in such a way that there are two end contractions. The water in the channel is 3.75 ft. deep. Compute the discharge.

266. A rectangular channel 15 ft. wide carries 75 c.f.s. It is desired to construct a sharp-crested weir entirely across the discharge end which will maintain a depth of 4 ft. upstream from the weir. Determine the height of weir.

267. A rectangular channel 25 ft. wide carries 150 c.f.s. and a sharp-crested weir 3 ft. high extends across the outlet end. How deep is the water a short distance upstream from the weir?

131. Notched Weirs.—Rectangular weirs having a re. short crest and also a wide variety of weirs having openings other than rectangular are classified as notches. The commonest of these is the V-notch weir, illustrated in Figs. 169*e* and 172.

The usual V-notch weir is sharp-edged and is located so that the nappe is completely contracted on the two sides and at the surface. The nappe is triangular at the plane of the weir, chang-

[1] See Kauffman, *op. cit.*, vol. II, p. 47.

ing to a crescentlike cross section in a short distance and finally, if allowed to fall far enough, to a nearly circular jet. Any dimension of the nappe is proportional to the head H and the area of any cross section is therefore proportional to H^2. At some section such as aa the pressure in the nappe is atmospheric and the velocity is fairly uniform over the cross section, the area of which, being proportional to H^2, can be expressed as K_2H^2. The vertical distance from the free surface to the mass center of the water

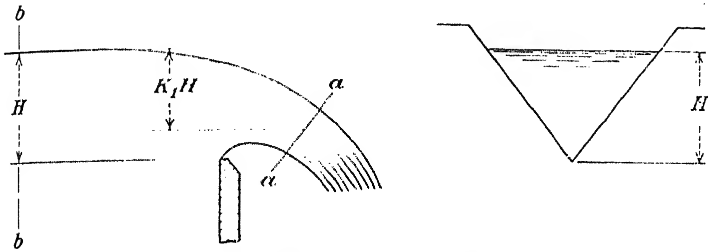


Fig. 172. V-notched weir.

flowing through section aa is proportional to H and can therefore be written K_1H.

The effect of velocity of approach is very small and may be neglected. Bernoulli's equation from any point in bb to section aa, where the average velocity is taken as U, is then

$$K_1H = \frac{U^2}{2g}$$

whence $U = \sqrt{2gK_1H}$ and the discharge is

$$Q = \sqrt{2gK_1H}\ K_2H^2 = \sqrt{2gK_1}K_2 H^{5/2}$$

The Q can be corrected for lost head and nonuniform velocity distribution by introducing a factor C'', after which

$$Q = C''\sqrt{2gK_1}K_2 H^{5/2} \tag{65}$$

Replacing the product $C''\sqrt{2gK_1}K_2$ by C, this becomes

$$Q = CH^{5/2} \tag{66}$$

Values of C have been determined by various investigators,[1]

[1] For a bibliography on the subject of notches, see F. W. Greve, Flow of Water through Circular, Parabolic, and Triangular Vertical Notch-Weirs, *Purdue Univ. Eng. Bull.* 40. This bulletin also gives results of researches by its author on notches other than 90-deg. notches.

most of whom have found the exponent of H to be slightly less than $5/2$. The C of Eq. (66) is not a constant but on the contrary varies slightly with H. The formula proposed by King,[1] based upon the calibration of a 90-deg. notch made of commercial steel plate, is

$$Q = 2.52H^{2.47} \tag{67}$$

The formula obtained by Barr[2] for a 90-deg. notch cut from a brass plate is

$$Q = 2.48H^{2.48} \tag{68}$$

Notches can be designed so that Q varies with any desired power of H within practical limits. If a notch is to be designed so that Q varies with H to the power N, the opening in the plane of the weir must be shaped so that the area varies with $H^{(N-\frac{1}{2})}$. Then the area at section aa is $K_2H^{(N-\frac{1}{2})}$ and

$$Q = C'\sqrt{2gK_1H}\,K_2H^{(N-\frac{1}{2})} = CH^N \tag{69}$$

When $N = 2$ the crest is in the form of a parabola with a vertical axis, the equation being $x^2 = 4aH$. Parabolic weirs have been calibrated by Greve,[3] who found the discharge to be as given by the formula

$$Q = 2.09a^{0.479}H^{1.99} \tag{70}$$

Figure 173 illustrates the form of weir opening for several values of N. The exponent N cannot be made less than unity and cannot be exactly unity since theoretically this would require infinite width at the bottom as indicated in Fig. 173a. A compromise form shown in Fig. 173b is used to regulate flow in channels in which it is desired to have a constant average velocity for all depths. If placed at the end of a channel of width b in which the velocity is V,

$$Q = bHV = CH$$

and

$$V = \frac{CH}{bH} = \frac{C}{b} \text{ (a constant)}$$

[1] KING, *op. cit.*

[2] BARR, JAMES, Flow of Water over Triangular Notches, *Engineering*, April, 1910.

[3] See footnote, p. 261.

For $N > \frac{5}{2}$ the outline of the crest is convex upward as in Fig. 173*f* and becomes increasingly so with higher powers of H.

Problem 268. A rectangular channel 5 ft. wide carries a maximum of 8 c.f.s. and a minimum of 5 c.f.s. The outlet is a 90-deg. V-notched weir. At what level should the notch be placed to make the maximum depth 4 feet? What is then the minimum depth?

269. A parabolic notched weir is to have a head of 1 ft. when Q is 2 c.f.s. What is the width of the opening at the level of the water surface?

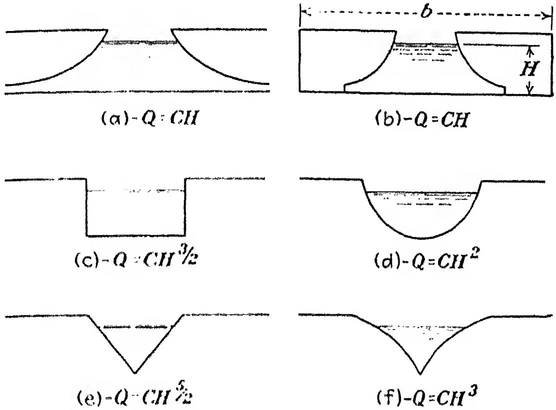


FIG. 173. Notched weirs for various powers of H.

132. Broad-crested Weirs. Weirs other than the sharp-crested type, for example, those shown in Figs. 169*c*, 169*d*, 174 and 175, are classed as broad-crested weirs. Such weirs are usually part of a dam or some other structure for the control of water and they are not well adapted to the measurement of flow.

If the upstream edge of the crest is sharp and the nappe leaps clear of the downstream edge, as in Fig. 174*a*, the discharge at a given head is practically the same as for a sharp-crested weir. If the thickness of the weir is more than about $0.47H$, the falling nappe does not clear the weir and the flow is as shown in Fig. 174*b*.

Figure 174*c* shows a section with a rounded upstream corner and a gently sloping top face. If the rounding of the edge is sufficient to eliminate contraction and if the slope is sufficient to maintain flow at critical depth, the flow is controlled by the critical-depth conditions shown in Fig. 166 and discussed in Art. 127. Neglecting

velocity of approach, it was there shown that $d = \frac{2}{3}H$ and $h = H/3$, whence the theoretical q per unit width is

$$q = (\text{area})\ (\text{velocity}) = \left(\frac{2}{3}H\right)\sqrt{2g\frac{H}{3}} = 3.09H^{3/2} \tag{71}$$

It has been shown experimentally that a good approximation to the actual discharge is given by the equation

$$q = 3.03H^{3/2} \tag{72}$$

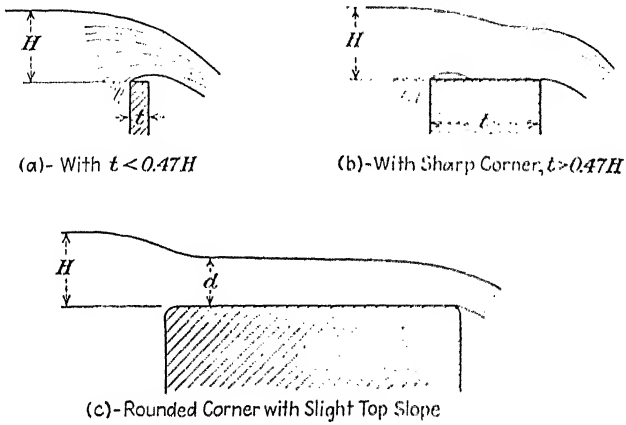


FIG. 174.—Broad-crested weirs.

The discharge over broad-crested weirs is usually computed by the basic formula

$$Q = CLH^{3/2} \tag{73}$$

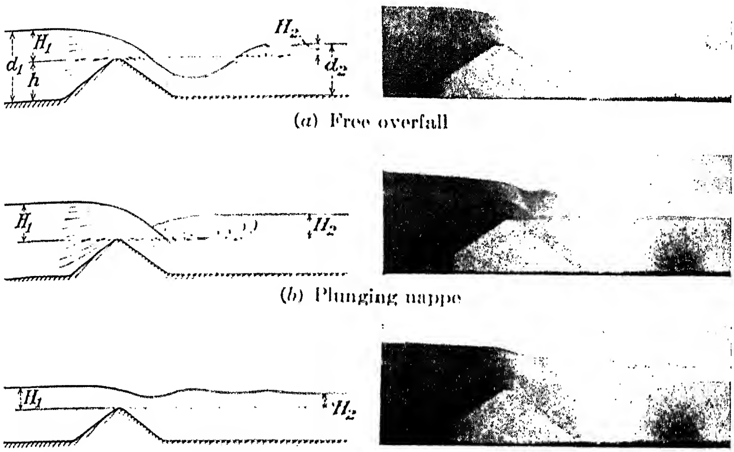
The coefficient C is not exactly constant and may vary considerably with H. The use of the formula therefore depends upon tables or graphs of C determined by experiment.[1]

Several experimenters have developed formulas, each of which applies only to a particular form of weir.

Problem 270. A weir of the form shown in Fig. 169c is 75 ft. long and discharges 1400 c.f.s. when H is 3 ft. Compute C. What would H be for the weir of Fig. 174c for the same length and discharge?

[1] For values of C consult Robert E. Horton, Weir Experiments, Coefficients and Formulas, *U. S. Geol. Survey Water Supply and Irrigation Paper* 200, and J. S. Woodburn, Tests on Broad-crested Weirs, *Trans. A.S.C.E.*, vol. 96, 1932.

133. Submerged Weirs.—Submerged weirs are those which are so situated that the downstream water level is higher than the crest. The free surface affected by a submerged weir takes one of the three forms shown in Fig. 175 if the depth of the upstream channel is greater than critical depth. When a distinct hydraulic jump is formed as in Fig. 175*a*, the depth of the water downstream from the weir has no effect on the discharge, even when the downstream surface is higher than the weir. When no jump is formed, as in Fig. 175*b* and *c*, the effect of submergence on discharge is very small for $H_2 < 0.5H_1$.



(*a*) Free overfall

(*b*) Plunging nappe

(*c*) Wave nappe

FIG. 175. Submerged weirs.

Up to a certain value of the ratio H_2/H_1 the nappe for a given weir is of the plunging type in Fig. 175*b*, while above this critical value the surface is of the wave form shown in Fig. 175*c* and the nappe does not plunge. These changes in form were recorded by Bazin and have been studied by many later experimenters. In experiments with a weir model of the form shown in Fig. 176, Keutner[1] found the flow to be of the wave type only for values of $H_2/H_1 > 0.85$. This ratio is not necessarily valid for weirs of different height or form. Keutner also found that the dis-

[1] KEUTNER, C., Neues Berechnungsverfahren für den Abfluss an Wehren aus der Geschwindigkeitsverteilung des Wassers über der Wehrkrone, *Die Bautechnik*, p. 575, 1929.

charge could be represented by a formula of the type used in free overfall for values of $H_2/H_1 < 0.775$. For greater degrees of submergence it was necessary to use a formula including a function of H_2/H_1, in which formula the coefficients were quite different for the two types of surface.

A rational formula for discharge over a submerged weir must include h, the height of the weir; otherwise it will give erroneous results for some cases. Probably no formula is valid if $H_1 > \frac{2}{3}h$. When H_1 is large in proportion to h there is very little disturbance of the surface, the presence of the weir being indicated only by a slight depression directly above it.

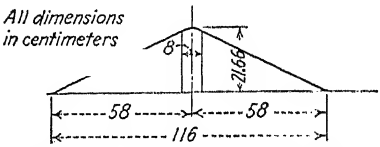


FIG. 176.—Submerged weir model.

The formula by Bazin for discharge over a submerged sharp-crested weir extending across a rectangular channel is

$$Q = LH_1^{3/2}\left(3.248 + \frac{0.079}{H_1}\right)\left(1 + 0.55\frac{H_1^2}{d_1^2}\right) \times \\ \times \left(1.05 + 0.21\frac{H_2}{h}\right)\sqrt[3]{\frac{H_1 - H_2}{H_1}} \qquad (74)$$

For depths less than critical, that is, for shooting flow, the effect of a low obstruction or weir is to produce a swell such as that shown in Fig. 177*a*. The free surface on the downstream face of the obstruction is somewhat disturbed because the stream is expanding to assume a greater depth. The new depth is necessarily greater because there has been a loss of energy and a corresponding reduction in velocity. That such an increase of depth must accompany a loss of energy in shooting flow is evident from an inspection of Fig. 160 or 165*c*. If the obstruction is made sufficiently high, the continuity of the surface is broken and a hydraulic jump is formed upstream from the weir as in Fig. 177*b*.

Problem 271. A sharp-crested weir 3 ft. high extends across a rectangular channel 20 ft. wide. The water upstream from the weir is 4 ft. deep. Compute the discharge when the depth downstream from the weir is (*a*) 2 ft., (*b*) 3.5 ft.

134. Critical-depth Meter.—The weir as a device for measuring flow is open to the objections that the formula is complicated by the correction for velocity of approach and also that considerable

fall must be allowed. These two objections are partially eliminated by making use of a combination of the phenomena of critical depth and hydraulic jump as sketched in Fig. 178.


(a)


(b)

FIG. 177. Low weirs with shooting flow, $d_1 < d_c$.

If the obstruction is well proportioned, the critical depth will exist near but not necessarily at the crest. Bernoulli's equation, written from any point in section 1 to any point at the critical

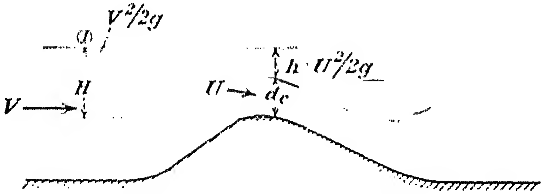


FIG. 178. Critical-depth meter.

section, where the average velocities are V and U, respectively, reduces to

$$\frac{V^2}{2g} + H = \frac{U^2}{2g} + d_c$$

whence

$$U = \sqrt{2g\left(H + \frac{V^2}{2g} - d_c\right)} = \sqrt{2gh}$$

and the discharge per unit width of channel, the product of velocity U and the area d_c, is

$$q = d_c\sqrt{2gh} \tag{75}$$

It was shown in Art. 124 that at critical depth the velocity head is half the depth or

$$\frac{U^2}{2g} = h = \frac{d_c}{2} \tag{76}$$

and, combining this with Eq. (75),

$$q = \sqrt{gd_c^3} \tag{77}$$

This is a simple rational equation for q which requires no correction for velocity of approach. Its use requires only one measurement, d_c.

The above theory is inaccurate in that V and U have been assumed uniform, the pressure is assumed to be static and losses have been neglected. It is also questionable because d_c is taken as a vertical linear distance which cannot possibly be at right angles to all stream tubes since they are not parallel. Moreover, the critical depth does not occur at the same horizontal position for different depths. This fact was substantiated by Woodburn and Webb.[1]

The device can be used by establishing a surface gage for measuring d at a point near the mean position of critical depth. The equation can be written

$$q = C\sqrt{gd^3} \tag{78}$$

the coefficient C being determined by calibration. It is not a constant but varies with d and will be only slightly less than unity.

135. Use of Weirs.—A weir intended to measure flow should be calibrated in the exact position in which it is to be used. After calibration the working conditions should not be altered. The use of any weir formula for the accurate computation of discharge depends not only upon the formula but also on the almost exact duplication of the conditions for which the formula was developed. The conditions to be duplicated include the material of the weir

[1] See footnote, p. 264.

plate, the sharpness of the edge, the cleanliness of the weir, the detailed nature of the flow in the approach channel and the method of measuring the head.

Many experiments have shown that roughness of the weir plate increases discharge for a given head. For example, weirs of glass, brass and steel have different coefficients, that for steel being greatest and that for glass smallest. Grease on any of these surfaces affects the shape of the nappe and the discharge.

The manner in which water approaches the weir has an important effect on the discharge, but the details of flow are difficult to duplicate in another channel. Moreover, a given flow regime in a channel cannot always be reproduced in another series of tests and may even change suddenly during a test without any apparent cause. It has been found that a weir with end contractions or one inclined downstream tends to promote stable conditions in the channel of approach.[1]

The place and manner in which the head is measured are other factors on which the accuracy of weir measurements depends. For exact similitude the head would have to be measured at different places for different heads and heights of weir. The practice is to make a connection to the channel well upstream so as to avoid the influence of surface contraction. A pipe leads from this connection to a container in which the gage is installed.

If the weir is so situated that the space under the nappe is closed, it is necessary to connect this space with the atmosphere. If the nappe is not aerated in this way, the moving water draws the air out and a partial vacuum exists under the nappe, which is depressed or which may cling to the face of the weir.

136. Transitions in Channels. A change in velocity or velocity distribution incident to a change of form or size in a channel always involves some loss of head. Even if the loss could be avoided there would be a change in elevation of the free surface equal to the increment of velocity head. Figure 179 shows the plan and profile of a reduction and of an enlargement typical of the case in which the flow of a canal is carried for some distance by a flume of smaller cross section, with a transition at each end of the flume.

[1] HAILER, R., Sources of Error in Weir Measurements, *Trans. Hydraulic Inst. Munich Tech. Univ., Bull.* 3, ed. by D. Thoma, translated by K. C. Reynolds, A.S.M.E., 1935.

The Bernoulli equation from point 1 to 2 for both reduction and enlargement is

$$\frac{V_1^2}{2g} + z = \frac{V_2^2}{2g} + \text{loss}$$

and

$$= \left(\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right) + \text{loss}.$$

In the enlargement z will usually be negative; in other words, the free surface will rise.

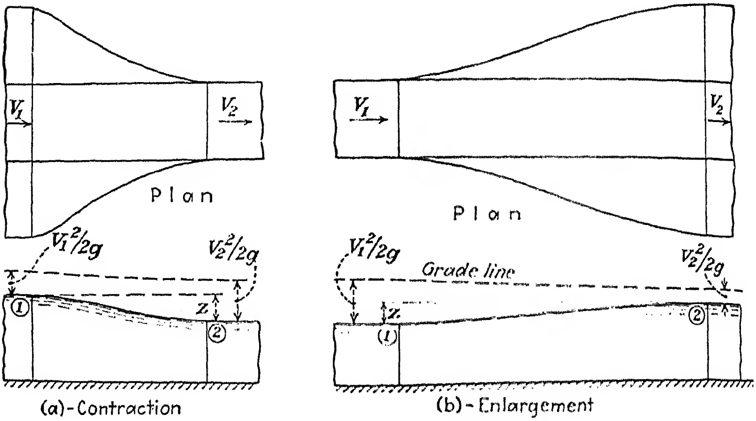


FIG. 179.—Channel transitions.

According to Hinds[1] the losses may be readily reduced by good design to $0.1\left(\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right)$ for a reduction or $0.3\left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right)$ for an enlargement.

The conversion of the stream from one form to a new one usually takes place in a transition structure. The reducing transition will be satisfactory if its shape is such that the flow follows all the walls and if it provides for the necessary drop in the free surface. The purpose of the enlargement structure is to expand the stream so that it has a distribution of velocity nearly as good as will exist in the canal and to do this with a minimum of lost energy. This result is usually obtained by using a long tapering structure designed to produce a smooth free surface tangent to

[1] HINDS, JULIAN, The Hydraulic Design of Flume and Siphon Transitions, *Trans. A.S.C.E.*, 1928, p. 1423.

that of the channels it joins. Even very long tapering enlargements are not always successful because there is a tendency for the stream to separate from one side of the transition and to continue at high velocity along the other side well into the downstream canal. Tests of models by Benson[1] indicate that the redistribution of velocity can be accomplished in a very short distance by placing a submerged hump or obstruction as shown in Fig. 180. This hump distributes the flow and nearly eliminates the eddies at the sides of the channel. The structure is short, its shape is simple, all surfaces being planes, and it does not obstruct floating debris.

FIG. 180.—Transition model.

The function of the hump is to increase the depth and pressure at the center, thereby forcing the stream to expand and fill the transition. A similar effect has been produced by reducing the pressure at the side walls by means of suction tubes.[2]

Problem 272. A trapezoidal channel carrying 105 c.f.s. has a bottom width of 10 ft., a depth of 3.2 ft. and side slopes of 1.5 horizontal to 1 vertical. It discharges into a semicircular flume 8 ft. in diameter. If the loss in the transition is two-tenths of the velocity head in the flume, what is the elevation of the bottom of the flume relative to the bottom of the channel?

273. A channel takes water from a pond at a velocity of 4 ft. per sec. If the channel is 3 ft. deep, what is the elevation of its bottom relative to the pond, the loss being two-tenths of the velocity head?

137. Nonstatic Pressure. Throughout this chapter the Bernoulli constant and the expression for specific energy have been written with the assumption of static distribution of pressure; that is, the pressure at a given depth in the moving liquid has been assumed to be the same as in a liquid at rest. This assumption is permissible only when the motion is nearly linear and for slopes such that the cosine of the slope angle is nearly 1.

[1] BENSON, M. H., Model Tests of Outlet Transitions, *Civil Eng.*, vol. 6, p. 760, 1936.

[2] See P. P. Ewald, T. Pöschl and L. Prandtl, "The Physics of Solids and Fluids," p. 288, Blackie & Son, Ltd., London, 1930.

Figure 181 shows a case of uniform flow with a steep slope, the thickness t of the stream being materially different from the vertical distance eb. The pressure at the bottom can be computed by considering the forces on the prism db. Letting its end area be A and its weight wtA, the component of weight normal to the bottom is

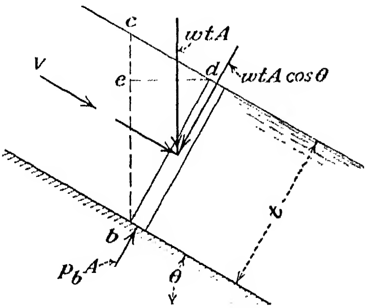


FIG. 181.—Uniform flow on a steep slope.

$$wtA \cos \theta = p_bA$$

and

$$p_b = w(t \cos \theta)$$

from which it appears that the effective pressure head on the bottom is $t \cos \theta$, which is distance eb in the figure.

When the path of the stream is a vertical curve the mass has a centripetal acceleration toward the center of curvature. The force necessary to produce this acceleration is such that $p_b < wd$

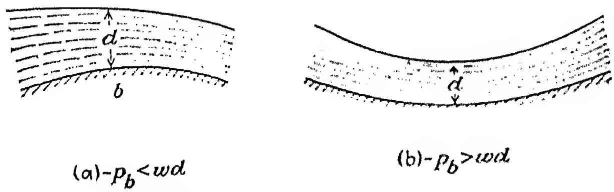


FIG. 182.—Effect of curvature on pressure.

when the center of curvature is below the stream (Fig. 182*a*) and $p_b > wd$ when it is above (Fig. 182*b*).

General Problems

274. A canal in earth has a bottom width equal to four times the depth and side slopes of 1 to 1. It is to carry 240 c.f.s. at a velocity of 3 ft. per sec. What slope is required when the canal is in good condition? With the same conditions, what is the slope of a semicircular canal with the same n?

275. Determine the arc of the circumference of a circular sewer which is wetted when the hydraulic radius is maximum. What is the corresponding depth in terms of the diameter?

276. A semicircular concrete flume 6 ft. in diameter has a slope of 0.003 and the flow is uniform. It discharges into a trapezoidal section with side slopes of 1 to 1, bottom width of 7 ft. and a depth of 3 ft. If the loss in the transition is four-tenths of the kinetic energy in the flume, what should be the relative elevations of the bottoms of the flume and the channel?

277. A channel has 1 to 1 side slopes and a bottom width of four times the depth. Find the critical depth when Q is 60 c.f.s. What is the critical velocity?

278. Water is flowing 1 ft. deep in a rectangular concrete channel at a velocity of 12 ft. per sec. Can a jump be formed? What is the depth downstream from the jump? If the channel is 14 ft. wide and the flow is uniform, what is the slope upstream from the jump?

279. A rectangular channel 12 ft. wide carries 144 c.f.s. A hydraulic jump is formed with the water 4 ft. deep on the downstream side. What is the depth upstream and how much head is dissipated?

280. The crest of a critical depth meter is 3 ft. higher than the bottom of a rectangular channel 20 ft. wide. Estimate the depth upstream from the meter when Q is 80 c.f.s

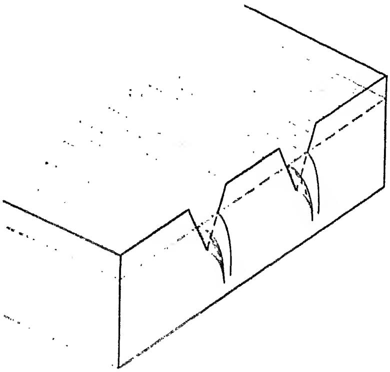


FIG. 183.

281. The flow in the channel of Fig. 183 varies from 8 to 30 c.f.s. and it is desired to regulate the depth by installing 90-deg. V-notch weirs at the end. How many weirs are needed to limit the variation in depth to 0.2 feet? What will be the maximum H?

282. The flow in a channel varies from 12 to 19 c.f.s. It is desired to discharge not less than 9 c.f.s. or more than 11 c.f.s. over a 90-deg. V-notch weir into one channel, while the remainder goes over a sharp-crested rectangular weir. Find the length of the rectangular weir and the maximum head on both weirs.

283. The bottom of a rectangular planked channel 20 ft. wide is 4 ft. below the surface of a pond from which it takes water. What is the maximum discharge that it can take from the pond and what minimum slope of channel is required?

284. A trapezoidal channel in the form of a half hexagon carries 100 c.f.s. a distance of 2000 ft. with a fall of 1.6 ft. What is the area of the cross section? What area is required with the same side slopes but with the bottom width four times the depth? In both cases $n = 0.025$.

285. A trapezoidal channel with a bottom width of 12 ft. and side slopes of 2 horizontal to 1 vertical carries 140 c.f.s. The depth at the end is to be

maintained at 5 ft. by a contracted weir 14 ft. long. Find the height of the weir.

286. A rectangular channel 40 ft. wide is 4 ft. deep and the flow is uniform. It has a slope of 0.004 and Kutter's n is 0.025. A weir is built in the channel which makes the depth 8 ft. just upstream from the weir. Compute the new depth at points 200, 400 and 600 ft. upstream.

287. At the edge of an abrupt waterfall the thickness of the overfalling nappe is 2 ft. What is the depth about 25 ft. upstream? Compute the discharge per unit width.

288. A rectangular concrete channel 10 ft. wide carries 20 c.f.s. in uniform flow. What is the critical slope? If the velocity is twice critical, to what height might the water jump and what is the specific energy before and after the jump? What head is lost?

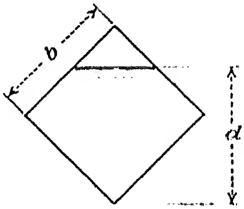


FIG. 184.

289. A square conduit is in the position shown in Fig. 184. Find d in terms of b for maximum velocity and for maximum discharge, assuming Manning's formula to hold.

CHAPTER XI

FLOW THROUGH ORIFICES AND TUBES

138. Flow through a Small Opening.—An expression for the velocity of discharge through a small opening is obtained by applying the principle of conservation of energy as embodied in the Bernoulli equation. The container of liquid in Fig. 185 has a small opening shaped so that the emerging particles move in parallel paths. The area of the free surface is A_s and the cross section of the issuing jet is A. The jet is discharging into atmosphere and the pressure throughout its cross section can therefore be assumed to be atmospheric. The pressure at a point immediately inside the opening is more than atmospheric, is less than the static pressure due to the head of liquid above and becomes nearly equal to that static pressure as a streamline is followed from the orifice back to regions of lower velocity.

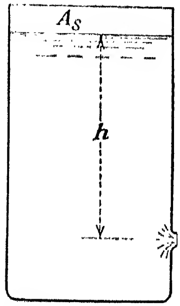


FIG. 185.—Velocity of efflux.

Writing Bernoulli's equation between the free surface, which, because of the discharge, is falling with a velocity V_a, and the center of the jet, which has a velocity of efflux u,

$$\frac{V_a^2}{2g} + \frac{p_a}{w} + h = \frac{u^2}{2g} + \frac{p_a}{w}$$

and

$$u = \sqrt{2g\left(\frac{V_a^2}{2g} + h\right)} = \sqrt{2gH} \tag{1}$$

in which H represents the total head or effective head. When the ratio of A_s to A is large, the term $V_a^2/2g$ is relatively small and can be neglected, in which case

$$u = \sqrt{2gh}$$

From this equation it appears that the velocity of efflux under a static head h is the same as for a body falling freely from rest

through a vertical distance h. This fact is known as Torricelli's theorem. The velocity may be referred to as the Torricelli velocity to differentiate it from the actual velocity, which is slightly smaller. Then

$$u_T = \sqrt{2gh} \tag{2}$$

The quantity flowing based on the Torricelli velocity is

$$Q_T = Au_T = A\sqrt{2gh} \tag{3a}$$

or, using the total head,

$$Q_T = A\sqrt{2gH} \tag{3b}$$

139. Effective Head on Small Openings. There are numerous cases in which the application of Torricelli's theorem is more

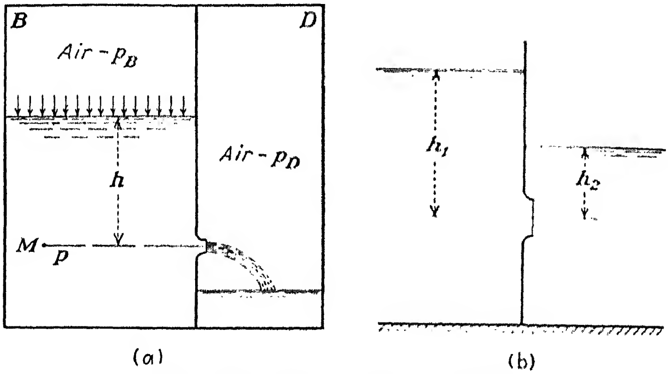


FIG. 186.—Effective head on orifices.

complicated than in the one just discussed. For example, Fig. 186a shows a case in which the upstream free surface of area A_s is subjected to a pressure p_B while the opening at a distance h below the free surface permits discharge into space D, in which the pressure is p_D. Writing Bernoulli's equation from the free surface to a point in the jet just outside the opening, where the velocity is u_T and the pressure is p_D,

$$\frac{V_a^2}{2g} + \frac{p_B}{w} + h = \frac{u_T^2}{2g} + \frac{p_D}{w}$$

from which

$$u_T = \sqrt{2g}\sqrt{\frac{V_a^2}{2g} + \left(\frac{p_B}{w} - \frac{p_D}{w}\right) + h} = \sqrt{2gH} \tag{4}$$

The effective head H is the quantity under the second radical. By a similar process an expression for effective head can be obtained for any case.

Point M in Fig. 186*a* is at the level of the opening and far enough removed so the pressure p at M is little affected by the presence of the opening. Neglecting $V_M^2/2g$, the Bernoulli equation from M to the jet is

$$\frac{p}{w} = \frac{u_T^2}{2g} + \frac{p_D}{w}$$

and

$$u_T = \sqrt{2g\left(\frac{p}{w} - \frac{p_D}{w}\right)} = \sqrt{2gH} \tag{5}$$

which shows that the effective head H is the difference in pressure heads on the two sides of the opening. This is true irrespective of the manner in which the pressure is maintained.

An orifice in the setting shown in Fig. 186*b* is said to be submerged. The effective head is $h_1 - h_2$ and the velocity is $u_T = \sqrt{2g(h_1 - h_2)} = \sqrt{2gH}$.

Problem 290. In Fig. 186*a* the pressure in space D is 12 lb. per sq. in. abs., in space B it is 4 lb. per sq. in. gage, $h = 12$ ft. and the water in D stands 5 ft. above the opening. (*a*) Compute the velocity of the jet. (*b*) Assuming p_B, p_D, h and the depth to be as above, compute the velocity of the jet if the specific gravity of the liquid is 0.7.

291. In Fig. 186*b*, $h_1 = 10$ ft. and $h_2 = 4$ ft. Compute the velocity of the jet under these conditions. Find the value of h_1 which will produce a velocity of 24 ft. per sec. in the jet, h_2 being unchanged.

140. Effective Head on a Large Opening.—A special problem presents itself when the head on an opening in a vertical plane is relatively small, that is, when the head is only a few times the vertical dimension of the opening. Theoretically the velocity through a horizontal element of area of such an opening is $u = \sqrt{2gy}$ where y is the effective head on the element. The average value of $\sqrt{2gy}$ is not exactly equal to its value for the center of the opening and it may therefore be necessary to use other means in finding the average velocity. In the following development both y and H are measured from the energy gradient. Thus H is the total effective head at the center of the opening.

Figure 187 shows a large rectangular opening so shaped as to discharge a full-sized jet. The Torricelli velocity through any element of area is

$$u_T = \sqrt{2gy} \tag{6}$$

The quantity flowing through the element of area $dA = b\,dy$ is

$$dQ_T = u_T\,dA = b\sqrt{2gy}\,dy \tag{7}$$

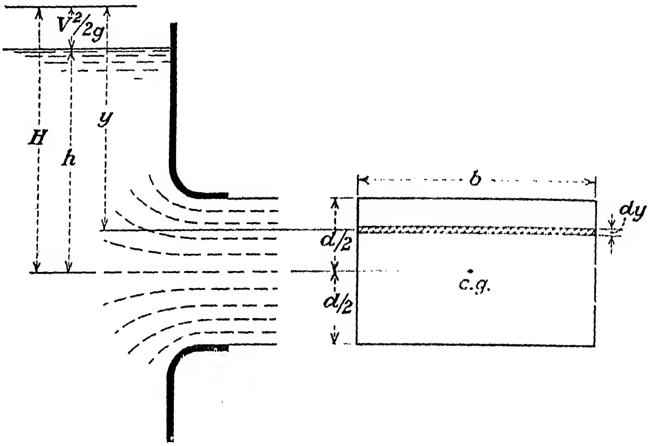


FIG. 187.—Large orifice.

and, integrating over the entire area,

$$Q_T = b\sqrt{2g}\int_{\left(H-\frac{d}{2}\right)}^{\left(H+\frac{d}{2}\right)} y^{1/2}\,dy = \tfrac{2}{3}b\sqrt{2g}\left(y^{3/2}\right)_{\left(H-\frac{d}{2}\right)}^{\left(H+\frac{d}{2}\right)}$$

or

$$Q_T = \tfrac{2}{3}b\sqrt{2g}\left[\left(H + \frac{d}{2}\right)^{3/2} - \left(H - \frac{d}{2}\right)^{3/2}\right] \tag{8}$$

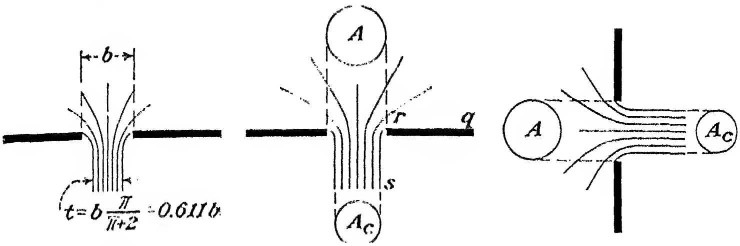
After expanding the binomials, dropping all terms except the first two and dividing out $\sqrt{H}$ and d, this becomes

$$Q_T = bd\sqrt{2gH}\left(1 - \frac{1}{96}\frac{d^2}{H^2}\right) \tag{9}$$

Since the product bd is the area A, the value of Q_T from Eq. (9) is equal to that of Eq. (3) multiplied by the quantity in brackets. The bracketed expression equals 0.989 for $H/d = 1$, 0.997 for

$H/d = 2$ and 0.999 when $H/d = 3$. From these values it appears that there can be only a small error in using Eq. (3) and the effective head is, for all practical purposes, equal to the head on the centroid of the opening. The error is even less for a circular opening. When the head is small enough to require correction, a vortex extending from the free surface into the jet will form and this will vitiate any formula, however correct in theory.

141. Contraction of Jets. The openings shown in Figs. 185 to 187 are shaped so that the issuing jet has a cross-sectional area equal to the area of the opening. The approach to the opening is from every part of the container and the necessary change in direction is completed within the container so that all particles



(a)-Horizontal Slot (b)-Circular Orifice in Horizontal Plate (c)-Circular Orifice in Vertical Plate

FIG. 188. Contraction of jets.

move in parallel paths as they pass through the plane of the opening.

Figure 188 shows several openings which are usually designated as sharp-edged orifices. The inner corner of such an orifice may be either a right angle or any acute angle. The approach is again from every part of the interior but the change in direction of the streamlines is not completed within and it continues on the downstream side. The streamline qrs, for example, bends through an arc of 90 deg. downstream from the plane of the opening. As a result of this necessary change in direction, the jet is smaller than the opening and it is said to be contracted. The section at which this contraction is complete is called the vena contracta. Here the streamlines are parallel and the pressure throughout this cross section is considered to be the same as in the surrounding medium, which is usually the atmosphere.

In the case of flow through a long sharp-edged slot, the cross section of which is shown in Fig. 188a, it can be shown by methods

of classical hydrodynamics[1] dealing with an ideal fluid that the thickness t of the sheet of water is $\frac{\pi}{\pi + 2}$ times the width b of the opening. Theoretically this contraction is completed at infinite distance from the orifice but practically it is complete at a distance of about one-half b or one-half the diameter D in the circular orifice.

The ratio of the cross section A_c at the vena contracta to the area A of the orifice is the coefficient of contraction C_c. Thus

$$A_c = C_c A \tag{10}$$

The theoretical value of C_c for a long sharp-edged slot is $\frac{\pi}{\pi + 2} =$ 0.611, which is also very near the value of C_c for flow of water through sharp-edged orifices with D equal to or larger than 2.5 in. The value of C_c is not known to be less than 0.611 for a circular orifice. Values of C_c for water obtained experimentally by Smith and Walker[2] are given in Table VIII below. Inspection of this table shows that C_c decreases with head up to about 60 ft. and with diameter up to 2.5 in. and tends to become constant for larger heads or diameters. This is taken by some writers

TABLE VIII.—COEFFICIENT OF CONTRACTION C_c FOR CIRCULAR SHARP-EDGED ORIFICES*

Head d, ft.	Diameter, in.				
	0.75	1.0	1.5	2.0	2.5
1.0	0.688	0.657	0.626	0.619	0.615
2.0	0.680	0.652	0.624	0.617	0.614
4.0	0.671	0.644	0.621	0.617	0.614
6.0	0.669	0.642	0.620	0.617	0.614
8.0	0.668	0.640	0.620	0.617	0.614
10.0	0.667	0.639	0.620	0.616	0.614
20.0	0.667	0.639	0.618	0.616	0.613
40.0	0.666	0.639	0.617	0.615	0.613
60.0	0.666	0.639	0.617	0.615	0.613

* SMITH and WALKER, *op. cit.*, pp. 34–35.

[1] RAMSEY, A. S., "Treatise on Hydromechanics," part II, p. 132, George Bell & Sons, Ltd. London, 1920.

[2] SMITH, DEMSTER, and WILLIAM J. WALKER, Orifice Flow, *Proc. Inst. Mech. Eng.* (*London*), 1923, p. 23.

to indicate that perfect contraction is not to be had for low heads or for diameters less than 2.5 in., a view supported by the coeffi-

TABLE IX.—COEFFICIENT OF VELOCITY C_v FOR CIRCULAR SHARP-EDGED ORIFICES*

Head, ft.	Diameter, in. 0.75	1.0	1.5	2.0	2.5
1.0	0.954	0.962	0.973	0.980	0.987
2.0	0.957	0.966	0.980	0.984	0.990
4.0	0.956	0.973	0.983	0.984	0.990
6.0	0.954	0.975	0.984	0.984	0.990
8.0	0.951	0.977	0.985	0.984	0.990
10.0	0.953	0.977	0.985	0.986	0.990
20.0	0.953	0.978	0.988	0.986	0.993
40.0	0.954	0.978	0.990	0.988	0.993
60.0	0.954	0.979	0.990	0.988	0.993

* SMITH and WALKER, *op. cit.*, pp. 34–35.

TABLE X.—COEFFICIENT OF DISCHARGE C FOR CIRCULAR SHARP-EDGED ORIFICES†

Head, ft.	Diameter, in. 0.75	1.0	1.5	2.0	2.5
1.0	0.657	0.633	0.612	0.607	0.606
2.0	0.651	0.630	0.611		
4.0	0.641	0.627			
6.0	0.638	0.626			
8.0	0.635	0.626			
10.0	0.635	0.625			
20.0	0.635	0.625			
40.0	0.635	0.625			
60.0	0.634	0.624	0.611	0.607	0.606

† SMITH and WALKER, *op. cit.*, pp. 34–35.

cients obtained earlier by Judd and King[1] and Bilton.[2] The coefficient C_c appears to be nearly constant for any orifice with

[1] JUDD, HORACE, and ROY S. KING, Some Experiments on the Frictionless Orifice, *Eng. News*, Sept. 27, 1906.

[2] BILTON, H. J. L., Coefficients of Discharge through Circular Orifices, *Eng. News*, July 9, 1908.

a head more than 10 ft. and for the larger orifices at much lower heads.

142. Coefficients of Velocity and Discharge.—The velocity as given by Torricelli's theorem is never quite realized because the conversion of energy in a real fluid cannot be accomplished without loss. The average velocity $U_T = \sqrt{2gH}$ must be modified by a coefficient of velocity C_v so that the real velocity is

$$V = C_v U_T = C_v \sqrt{2gH} \tag{11}$$

This coefficient is nearly unity for water. It is about 0.95 for a ¾-in. sharp-edged circular orifice and increases to 0.99 for a diameter of 2.5 in. It is nearly constant for a given orifice, increasing slightly for high heads. Values of C_v for water are given in Table IX.

The discharge is the product of the real velocity and the area at the vena contracta, or

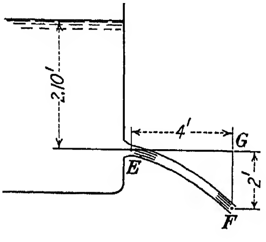
$$Q = A_c V = C_c A C_v \sqrt{2gH} \tag{12}$$

This may be written as

$$Q = CA\sqrt{2gH} \tag{13}$$

in which $C = C_c C_v$ is the coefficient of discharge, so called because it is the ratio between the actual Q as given by Eq. (12) or (13) and Q_T, uncorrected by contraction or velocity coefficients, as given by Eq. (3). For small circular orifices C decreases slightly with increasing heads and diameters, becoming nearly constant for diameters of 2.5 in. or more. Values of C for water are given in Table X. Fair approximate values to remember are $C = 0.60$, $C_v = 0.98$, $C_c = 0.611$.

Example.—A jet of water is discharged through a 1-in. diameter orifice under a head of 2.10 ft., the total discharge being 228 lb. in 90 sec. The jet is observed to pass through a point 2 ft. downward and 4 ft. away from the vena contracta. Compute C_c and C_v.



Solution.—Neglecting air resistance, the horizontal component of the jet velocity is unchanged, that is, it continues to be V, the actual velocity at the vena contracta. It falls with the acceleration of gravity. If a particle of water travels from E to F in t sec., the distances EG and GF in terms of t are

$$4 = Vt \qquad \text{and} \qquad 2 = \tfrac{1}{2}(32.2)t^2 = 16.1t^2$$

From the second equation $t = 0.352$ sec. and, substituting this in the first, $V = 11.36$ ft. per sec. The Torricelli velocity is

$$U_T = \sqrt{2gH} = \sqrt{64.4 \times 2.10} = 11.62 \text{ ft./sec.}$$

Then

$$C_v = \frac{V}{U_T} = \frac{11.36}{11.62} = 0.978$$

The discharge is $228 \div (62.4 \times 90) = 0.0406$ c.f.s. and, substituting this in the discharge equation, $Q = CA\sqrt{2gH}$,

$$0.0406 = C \cdot \frac{0.785}{144} \sqrt{64.4 \times 2.10}$$

and

$$C = 0.641$$

Then

$$C = C_c C_v$$

so that

$$C_c = \frac{C}{C_v} = 0.641 \div 0.978 = 0.655$$

143. Incomplete Contraction.—The extent of the contraction of a jet from an orifice is changed by any condition that alters the manner in which the fluid approaches the opening. The sharp-edged diaphragm orifice in a pipe line shown in Fig. 189 discharges a contracted jet which expands and again fills the pipe. The amount of contraction for a given d_1 diminishes with an increase in d, and C_c is a function of d/d_1, the ratio of diameters, or a function of A/A_1, the ratio of area of the opening to the cross-sectional area of the pipe. Values of C_c determined by Weisbach[1] for water follow.

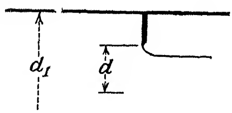


FIG. 189.—Diaphragm orifice.

A/A_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C_c	0.624	0.632	0.643	0.659	0.681	0.712	0.755	0.813	0.892	1.000

When an orifice is located near one or more side walls of a container, the approach of the fluid to the adjacent side of the orifice is somewhat restricted and the contraction of the jet is incomplete. If the orifice plate is very rough, the approach from

[1] See footnote, p. 215.

the sides is likewise restricted, the contraction is less and consequently the coefficient of discharge is increased.

Problem 292. Compute the diameter of a jet of water from a 2-in. circular sharp-edged orifice when the head is 10 ft. What are the exact velocity and discharge, using coefficients from the tables?

293. The velocity at the vena contracta of the jet of water from a 4-in. circular orifice is found to be 23.4 ft. per sec. when the head is 9 ft. and the discharge is 1.26 c.f.s. Compute C_v and C_c.

294. In Fig. 189 $d_1 = 12$ in. and $d = 6$ in. When Q is 2 c.f.s., what are the velocities of the water in the pipe and at the vena contracta? If C_v is 0.98, what is the effective head on the orifice?

144. Correction for Velocity of Approach. Velocity of approach may have considerable effect on the quantity discharged through

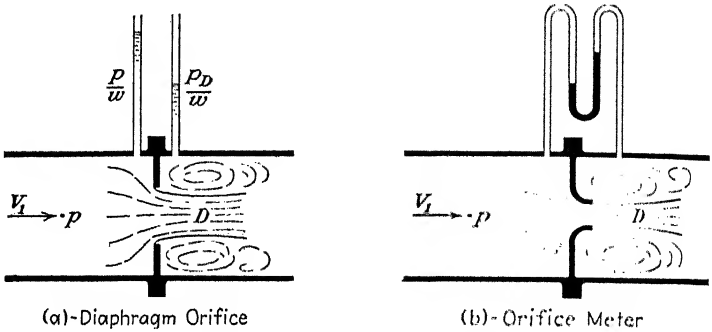


FIG. 190.—Orifices in pipe lines.

orifices such as those shown in Fig. 190. The effective head on such openings is

$$H = \left(\frac{p}{w} - \frac{p_D}{w} + \frac{V_1^2}{2g}\right) = \frac{\Delta p}{w} + \frac{V_1^2}{2g}$$

Neglecting velocity of approach, the rate of discharge can be expressed as

$$Q = CA\sqrt{2g\frac{\Delta p}{w}} \tag{14}$$

The coefficient of discharge C in Eq. (14) would vary greatly with head or Δp because it would have to correct for velocity of approach as well as for energy loss and contraction. A rational equation correcting for velocity of approach is often desirable. The velocity without correction for loss is

$$U_T = \sqrt{2g\left(\frac{\Delta p}{w} + \frac{V_1^2}{2g}\right)}$$

and the actual discharge is

$$Q = CAU_T$$

Substituting $V_1 = Q/A_1$ in the expression for U_T, the expression for Q becomes

$$Q = CA\sqrt{2g\left(\frac{\Delta p}{w} + \frac{Q^2}{2gA_1^2}\right)}$$

and, solving for Q,

$$Q = CA \quad \frac{2g\frac{\Delta p}{w}}{1 - C^2\left(\frac{A}{A_1}\right)^2} \tag{15}$$

A similar expression is obtained by writing

$$Q_T = A\sqrt{2g\left(\frac{\Delta p}{w} + \frac{V_1^2}{2g}\right)}$$

Then letting $V_1 = Q_T/A_1$ and solving for Q_T

$$Q_T = A \quad \frac{2g\frac{\Delta p}{w}}{1 - \left(\frac{A}{A_1}\right)^2}$$

Modifying this by the coefficient of discharge,

$$Q = CQ_T = CA\sqrt{\frac{2g\frac{\Delta p}{w}}{1 - \left(\frac{A}{A_1}\right)^2}} \tag{16}$$

Equations (15) and (16) become identical when C is nearly unity, that is, when the contraction is small. For cases in which the contraction is large, the values of C in Eqs. (15) and (16) will differ. In any event they must be determined by experiments, and Eq. (16) lends itself more readily to such determinations.

Fair values of C for water flowing through the orifice of Fig. 190*a* can be obtained by taking the product of an estimated value of C_v, say 0.98, and the values of C_c given in Art. 143.

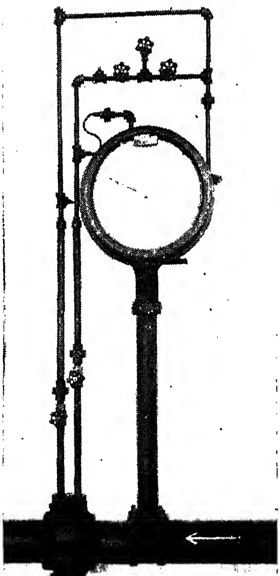


FIG. 191.—Orifice-meter installation. (*Courtesy of The Foxboro Company.*)

Orifices arranged as in Fig. 190 are commonly called orifice meters,[1] and a commercial orifice-meter installation is shown in Fig. 191. Equations (15) and (16) apply not only to those but to any case in which the velocity of approach is important, as, for example, the end orifice and the simple nozzle of Fig. 192.

In setting up the expressions for U_T and Q_T in this article it was assumed that the kinetic energy per pound of fluid is represented by $V_1^2/2g$. Since the velocity distribution upstream is nonuniform, the actual kinetic energy is much greater than $V_1^2/2g$, while the kinetic energy in the vena contracta where the velocity is quite uniform is nearly $V^2/2g$. This excess of actual kinetic energy over the assumed $V_1^2/2g$ results in some cases in very high experimental values of C_v or C, the former often closely approaching unity. The values of C depend to some extent upon the location of the pressure taps and in choosing coefficients the location of the taps must be considered.

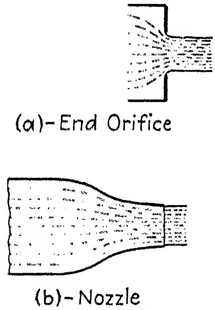


FIG. 192.—Free discharge.

Problem 295. Using $C_v = 0.98$ and Weisbach's coefficient of contraction, compute the discharge of water from the 6-in. orifice in Fig. 192*a* when the pressure in the 12-in. pipe is 10 lb. per sq. in. gage. What is the discharge under the same conditions if the liquid has a weight of 55 lb. per cu. ft.?

296. The orifice in the 12-in. pipe of Fig. 190*b* is 4 in. in diameter and discharges a jet of the same size. The coefficient of discharge is 0.94 and Q is 3 c.f.s. What is the difference between the pressure upstream and that in the jet?

[1] For further theory and coefficients on orifice meters, see Fluid Meters, Their Theory and Application, *A.S.M.E. Research Publ.*, 1931.

145. Loss of Head in Orifice Flow.—The Torricelli velocity with a total effective head of H is $U_T = \sqrt{2gH}$ but the real velocity in the jet is only

$$V = C_v \sqrt{2gH} \tag{17}$$

Writing Bernoulli's equation from a point upstream to the vena contracta and noting that the effective head H is the Bernoulli constant on the upstream side,

$$H = \frac{V^2}{2g} + \text{loss}$$

and

$$\text{loss} = H - \frac{V^2}{2g} \tag{18}$$

After eliminating either V or H by substituting its value as obtained from Eq. (17), the loss of head can be expressed as

$$\text{loss} = \left(\frac{1}{C_v^2} - 1\right)\frac{V^2}{2g} \tag{19}$$

or

$$\text{loss} = (1 - C_v^2)H \tag{20}$$

Equation (19) expresses the loss of head in terms of the actual velocity head after loss while Eq. (20) expresses it in terms of the total initial head. The dimension is a length, usually feet of head of the fluid in question.

Problem 297. Compute the head lost in a 2-in. orifice discharging 0.3 c.f.s. under a head of 9 ft. if C_c is 0.61. What is C_v?

298. A jet from a small orifice in a vertical plane falls vertically 2.58 ft. from the orifice while moving 6.4 ft. horizontally. The measured head on the orifice is 4.2 ft. Compute the velocity, the head lost and C_v.

146. Converging Orifices.—An orifice shaped so that the side walls converge will either reduce or entirely eliminate any contraction of the jet downstream from the plane of the opening.

With the shape shown in Fig. 193*a* the stream converges within and there is no contraction beyond the plane of the opening. The coefficients of velocity and discharge are equal and have a value of about 0.98.

With the conical form in Fig. 193*b* there is a slight contraction within the tube at M and a further contraction of the free jet

just outside of the cone, which latter contraction diminishes as the degree of convergence is reduced. The coefficient of velocity increases with θ and the maximum value of $C = C_c C_v$ is obtained when the angle is about 14 deg. It is then about 0.95.

It will be seen from Fig. 193*c* that even a slight rounding of the supposedly sharp edge of an orifice will have considerable effect on the size of the jet and will thereby increase the quantity discharged.

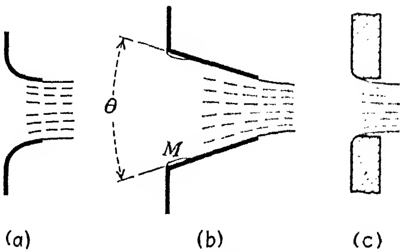


FIG. 193.—Converging orifices.

147. Short Tubes.—The term short tube is applied to orifices with downstream extensions not long enough to be classed as pipes. Such a tube is shown in Fig. 194. The orifice formulas are well adapted to the computation of discharge for such devices.

A standard short tube is a smooth tube with a sharp internal corner and a length equal to 2.5 diameters. Such a tube can flow with the jet remaining clear of the tube as in Fig. 194*a*, in which case the form of jet and the quantity discharged are the same as

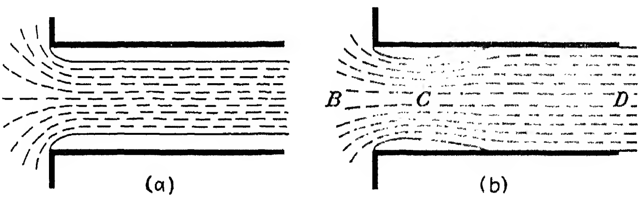


FIG. 194.—Short tubes.

for a sharp-edged orifice, or it can discharge a stream as large as the tube as in Fig. 194*b*. The latter type of flow is obtained by first stopping the tube and then permitting flow or by momentarily obstructing the flow of Fig. 194*a*, whereupon the tube begins to flow full. With the tube flowing full, the jet is not smooth and the flow is evidently turbulent.

With the tube flowing full, the jet is as large as the opening, $C_c = 1$ and $C = C_v$. The coefficient C is found experimentally to be about 0.82 for water. The loss of head computed by Eqs. (19) and (20) is

$$\left(\frac{1}{0.82^2} - 1\right)\frac{V^2}{2g} = 0.487\frac{V^2}{2g} \tag{21}$$

or

$$(1 - 0.82^2)H = 0.328H \tag{22}$$

This loss of head is many times that for the sharp-edged orifice, which is to be expected because the stream is caused to diverge.

By inspection it is evident that the pressure at the vena contracta within the tube is less than atmospheric and also that, if the tube is adequately vented to the air at C, it cannot flow full.

The stream within the tube is probably contracted as much as for the orifice and is probably contracted somewhat more for a given depth of water if the tube is less than 2.5 in. in diameter.

148. Limitation of Standard Short Tube.—The fact that the pressure at the vena contracta within the standard short tube is less than that at the end of the tube places certain limitations on the device. Let it first be assumed that $A_c/A = 0.62$. Then from the continuity equation $V = 0.62V_c$ and

$$\frac{V_c^2}{2g} = \frac{1}{0.62^2}\frac{V^2}{2g} = 2.60\frac{V^2}{2g}$$

The loss upstream from the vena contracta can be assumed to be about the same as for an orifice, that is, $0.04V_c^2/2g$, which is here equal to $(0.04)(2.60)\frac{V^2}{2g} = 0.104\frac{V^2}{2g}$. Now in Fig. 194*b* the loss between C and D is the loss from B to D less that from B to C, and from Eq. (21) it can be estimated as

$$\text{loss } (C \text{ to } D) = 0.487\frac{V^2}{2g} - 0.104\frac{V^2}{2g} = 0.383\frac{V^2}{2g}$$

Writing Bernoulli's equation between C and D and correcting for loss,

$$\frac{V_c^2}{2g} + \frac{p_c}{w} = \frac{V^2}{2g} + \frac{p_a}{w} + 0.383\frac{V^2}{2g} \tag{23}$$

After eliminating V_c and solving,

$$\frac{p_c}{w} = \frac{p_a}{w} - 1.22\frac{V^2}{2g} \tag{24}$$

or, since $V = 0.82\sqrt{2gH}$ and $V^2/2g = 0.672H$,

$$\frac{p_c}{w} = \frac{p_a}{w} - 0.82H \tag{25}$$

The fact that p_c never can be less than zero and practically cannot be less than the vapor pressure establishes the limiting head for which the tube can flow full. Above this limit the jet clears the tube as in Fig. 194*a*. The limiting head, neglecting vapor pressure, that is, assuming p_c to be zero, is

$$H = \frac{34}{0.82} = 41.5 \text{ ft.}$$

which agrees fairly well with experiments.

Problem 299. A standard short tube 2 in. in diameter is under a head of 16 ft. What are the discharge and lost head (*a*) when flowing full, (*b*) when flowing as an orifice? Compute the maximum discharge the tube can have when flowing full.

300. When a short tube 3 in. in diameter is flowing full under a head of 10 ft., what is the minimum pressure in the tube?

149. Re-entrant Tubes—Borda Mouthpiece. Openings in the form of tubes which extend from the wall of the container into the body of fluid, that is, re-entrant tubes, have a jet contracted somewhat more than that from the sharp-edged orifice in the plane of the wall. As the angle θ of Fig. 195*a* diminishes, the contraction increases. When the sides are parallel as in Fig. 195*b*, the coefficient of contraction is 0.5.

The tube of Fig. 195*b* with a sharp internal edge and with parallel sides about 2.5 diameters in length is known as a Borda mouthpiece and is of special interest because the coefficient of contraction of 0.5 can be obtained by the methods of classical hydrodynamics or by elementary mechanics, as is done in Art. 150. The coefficient of velocity is about 0.98 for water, making $C = 0.49$. Values of C_c for re-entrant tubes lie between 0.62 for the plane orifice and 0.5 for the tube with parallel sides. These values increase with θ, the angle of convergence.

150. Contraction of Jet in Borda Mouthpiece.—The inner end of a Borda mouthpiece is far enough from the wall of the container so that the presence of the opening has very little effect on the pressure at the wall. If it be assumed that the opening does not affect this pressure, the distribution of pressure on the wall

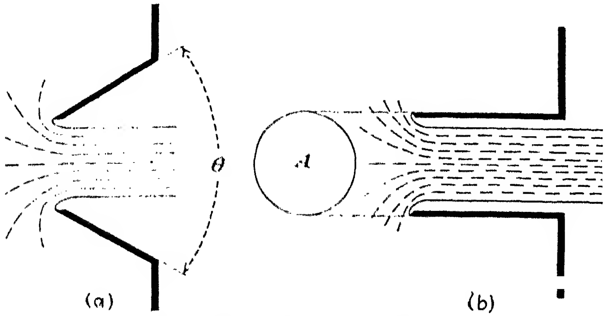


Fig. 195.—Re-entrant tubes.

having the opening and on the opposite wall is as shown in Fig. 196 by diagrams *abc* and *def*, respectively. There is an excess of pressure force, equal to whA, acting toward the left on the container or toward the right on the liquid. This effective force gives momentum to Q c.f.s. and equating force to change in momentum

$$whA = \frac{wQV}{g}$$

If C_v is taken as equal to unity, then $V = \sqrt{2gh}$ and $Q = A_c\sqrt{2gh}$. After substituting these in the above equation and solving

$$\frac{A_c}{A} = 0.5 = C_c$$

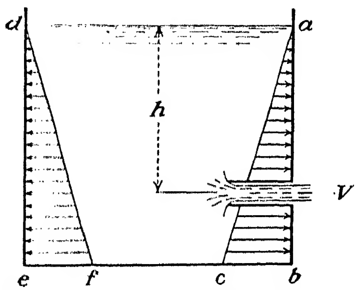


Fig. 196.—Pressure distribution with Borda mouthpiece.

This result has been closely approximated by experiments with water in which the actual discharge was measured and C_v was assumed to be 0.98.

The contraction of the jet within a Borda mouthpiece flowing full, as in Fig. 197, can also be investigated by elementary mechanics. It is again assumed that there is no loss upstream from the vena contracta and no change in pressure at the walls

of the container. The loss of head in the expanding stream is identical with that discussed in Art. 108. Making the appropriate substitution in Eq. (42) of that article

$$\text{Loss} = \left(1 - \frac{A_c}{A}\right)^2 \frac{V_c^2}{2g} = (1 - C_c)^2 \frac{V_c^2}{2g} = \frac{(1 - C_c)^2}{C_c^2} \frac{V^2}{2g}$$

Writing Bernoulli's equation from the free surface to the discharging jet and neglecting the small loss upstream from the

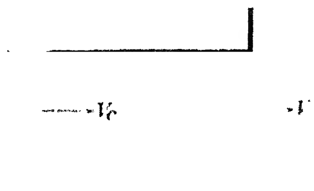


FIG. 197. Borda mouthpiece flowing full.

vena contracta,

$$h = \frac{V^2}{2g} + \frac{(1 - C_c)^2}{C_c^2} \frac{V^2}{2g}$$

and

$$V = \sqrt{2gh}\sqrt{\frac{C_c^2}{C_c^2 + (1 - C_c)^2}}$$

The change in momentum of the discharging water is again equated to the effective pressure force, whence

$$whA = \frac{wQV}{g} = \frac{wAV^2}{g}$$

After substituting the above expression for V and solving,

$$C_c = 0.5$$

This result is to be expected since it is based on the same assumptions as was C_c for free discharge. The expansion of the jet serves only to reduce the pressure and increase the effective head at the vena contracta.

151. Diverging Tubes.—If the sides of the tube shown in Fig. 198 do not diverge too rapidly and if the total divergence is not too great, the stream expands and fills the diverging portion of the tube.

The pressure at the throat of the tube is less than at any other point. It can be computed by writing Bernoulli's equation from the throat C to the discharge end. The pressure at D is atmospheric and, neglecting losses, which are usually large in an expanding stream, Bernoulli's equation from C to D is

$$\frac{V_c^2}{2g} + \frac{p_c}{w} = \frac{V^2}{2g} + \frac{p_a}{w}$$

The absolute pressure head at C is then

$$\frac{p_c}{w} = \frac{p_a}{w} - \left(\frac{V_c^2}{2g} - \frac{V^2}{2g}\right) \tag{26}$$

The actual velocity at D is $V = C_v\sqrt{2gH}$ and $C = C_v$, assuming no change in size of the jet beyond D. The total loss of head is $\left(\frac{1}{C_v^2} - 1\right)\frac{V^2}{2g}$. Most of the loss occurs downstream from the throat and, if it is assumed that all of it is downstream, the pressure head at C corrected for loss is

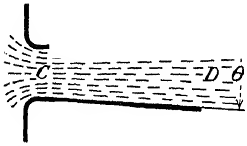


FIG. 198.—Diverging tube.

$$\frac{p_c}{w} = \frac{p_a}{w} - \left(\frac{V_c^2}{2g} - \frac{V^2}{2g}\right) + \left(\frac{1}{C_v^2} - 1\right)\frac{V^2}{2g} \tag{27}$$

The coefficient C varies with the size, shape and angle of the tube. Many experiments have been made on such tubes but the data are scattered and not readily correlated.

It is certain that C diminishes rapidly as the angle increases or as the length increases for a given angle. For example, Russell,[1] from experiments on a tube 1.22 in. in diameter and 6 in. long, finds $C = 0.83$ for $\theta = 5$ deg. and $C = 0.32$ for $\theta = 15$ deg. and, with the same throat and a length of 12 in., $C = 0.61$ for $\theta = 5$ deg. and 0.15 for $\theta = 15$ deg.

With increasing head the difference in velocity heads increases rapidly and p_c/w becomes small. Practically it cannot become zero owing to vapor pressure and separation of air. There is therefore a limiting head for each tube above which it cannot flow full.

[1] RUSSELL, GEORGE E., "Textbook on Hydraulics," p. 114, Henry Holt & Company, New York, 1934.

If the tube is considered to consist of the orifice of Fig. 193*a* with an added diverging portion, it will be seen that the orifice, which discharges against atmospheric pressure when alone, is discharging against a smaller pressure when fitted with the diverging section. The diverging section has then reduced the pressure at C. In other words, it has placed a suction or draft on the orifice and is therefore often called a draft tube.

Problem 301. A diverging tube is 3 in. in diameter at the throat and 4 in. at the end. Neglecting losses, find (*a*) the discharge of water under a head of 4 ft., (*b*) the maximum head for which it can flow full of cold water, (*c*) the maximum head for which it can flow full of water at 70°C.

302. A diverging tube is 3 in. in diameter at the throat and 4 in. at the end. The taper is such that $C = 0.6$. What is the discharge of water under a head of 4 ft. if the tube flows full? If the diverging portion is removed and C is then 0.97, what is the discharge? Compute the head lost for both cases.

152. Discharge under Falling Head.

In computing the time required to empty a vessel through an orifice, it is necessary to treat the head as a variable. Since the rate of discharge is proportional to the square root of head, the average head during the time of discharge is not to be used. It will be assumed that C is a constant over the range of head involved and that the rate of discharge at any time under the existing condition of unsteady flow is the same as if the flow were steady.

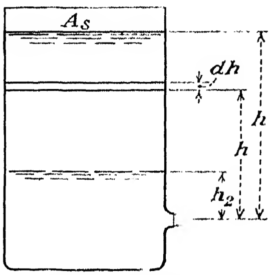


FIG. 199.—Falling head.

The container shown in Fig. 199 is being emptied through an orifice of area A. It has a free surface of area A_s which is to be lowered from a distance h_1 above the center of the orifice to a new level h_2. The rate of discharge at any time is $Q = CA\sqrt{2gh}$ and the volume discharged during an element of time dt is $Q\,dt = CA\sqrt{2gh}\,dt$. During this same time dt the free surface A_s is lowered and the head is reduced by a decrement dh. The volume discharged must equal the space vacated in the tank, whence

$$Q\,dt = CA\sqrt{2gh}\,dt = -A_s\,dh \tag{28}$$

Solving for dt and integrating,

$$= \int_0^t dt = \frac{-A_s}{CA\sqrt{2g}} \int_{h_1}^{h_2} h^{-1/2}\, dh$$

and

$$t = \frac{2A_s}{CA\sqrt{2g}}(h_1^{1/2} - h_2^{1/2}) \tag{29}$$

Equation (29) is useful if C is practically a constant. If A_s is not constant but can be expressed in terms of h, the time of discharge can be determined by substituting the expression for A_s in Eq. (28), solving for dt and integrating.

Problem 303. A tank 4 ft square has a sharp-edged orifice 2 in. in diameter in its side. If the water surface stands 9 ft. above the orifice, compute the time required to lower the water 4 ft.

304. Figure 200 shows a section of a tank which has vertical ends and a length of 20 ft. The orifice has an area of 24 sq. in. and $C = 0.6$. What time elapses while the surface is lowered 6 ft. from the position shown?

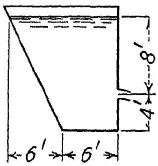


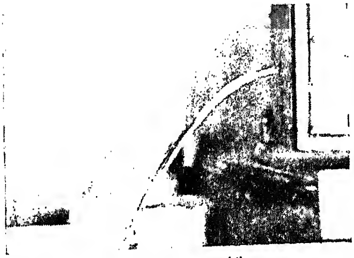
FIG. 200.

153. Inversion of Jets.—In the earlier articles of this chapter it has been assumed that the pressure at the vena contracta and points beyond on a jet is zero. This is not strictly true for smooth jets such as the one from a circular orifice because it is surrounded by a surface film. This film is in tension and the jet must therefore be under a small pressure. The streamlines in the top and bottom of a jet have slightly different velocities and tend to converge. This forces the film to be slightly elliptical instead of round; the former form being unstable, the film pulls itself and the jet back into a circular cross section. The transverse motion thus set up continues until the cross section is elliptical with the axis vertical. This action continues and the jet is alternately elliptical and round with the major axis of the ellipse turned through about 90 deg. in each half cycle of the motion. The slight rotation of the jet keeps the angle from being exactly 90 deg.

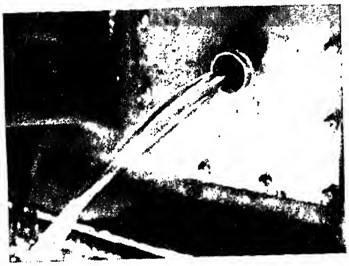
Figure 201 shows typical jets from sharp-edged orifices. The jet from the triangular orifice is nearly hexagonal at the vena contracta, takes a form of cross section having three lobes, one opposite each angle, and then completely inverts so that the lobes move through 180 deg. Meanwhile there is a slight rotation and the lobes become less distinct as the inversion is repeated.

The jet from the square orifice goes through a similar inversion, being first octagonal and then four-lobed in cross section. The jet from a rectangular opening has a similar form, the longest lobes being opposite the long sides.

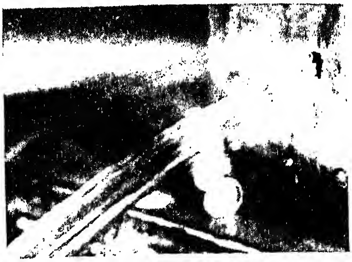
154. Effect of Viscosity on Velocity of Efflux.—The expression for velocity in a jet from an orifice was developed without consideration for density or viscosity. With no losses this velocity



(a) Circular orifice.



(b) Triangular orifice.



(c) Rectangular orifice.

FIG. 201.—Inversion of jets from sharp-edged orifices.

was found to be $U_T = \sqrt{2gH}$ and, corrected for losses, $V = C_v\sqrt{2gH}$, in which C_v, the coefficient of velocity, is nearly constant. The possible effect of density and viscosity may be studied by assuming the velocity to depend on ρ and μ as well as on g and H. The velocity is then a function of all four and might be expressed in the exponential form

$$V = Kg^aH^b\mu^c\rho^d \tag{30}$$

Keeping K dimensionless and substituting the fundamental dimensions M, L and T for the other quantities gives

$$\frac{L}{T} \approx \left(\frac{L}{T^2}\right)^a L^b \left(\frac{M}{LT}\right)^c \left(\frac{M}{L^3}\right)^d \tag{31}$$

For dimensional homogeneity it is necessary that

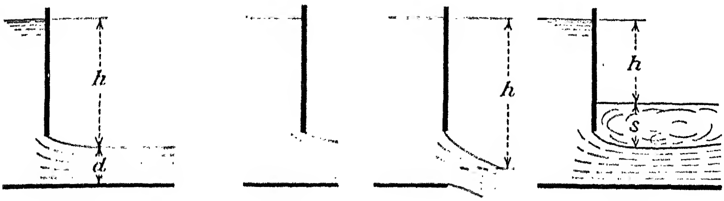
$$0 = c + d$$
$$1 = a + b - c - 3d$$
$$1 = 2a + c$$

Solving these equations for a, b and d, they are found to be $a = \frac{1-c}{2}$, $b = \frac{1-3c}{2}$ and $d = -c$. Then Eq. (30) becomes

$$V = Kg^{\frac{1-c}{2}} H^{\frac{1-3c}{2}} \frac{\mu^c}{\rho^c} \tag{32}$$

If it is true that V varies as $H^{\frac{1}{2}}$, then $b = \frac{1-3c}{2} = \frac{1}{2}$ and c is zero.

It follows then that, if V varies with the square root of H, the velocity is quite independent of both viscosity and density. Examination of Table IX, page 281, indicates a slight variation of C_v with head and it is therefore likely that viscosity does have some effect but a very small one.



(a) (b) (c) (d)

FIG. 202. Discharge of gates.

The above statements cannot be applied to tubes and nozzles because it is to be expected that viscosity effects will be introduced by any extension of an orifice downstream from the vena contracta.

155. Discharge of Gates. The orifice theory is applied to many types of openings and conduits. For example, the discharge of a culvert is often expressed as $Q = K\sqrt{H}$, and that through a sluice gate as $Q = KA\sqrt{2gH}$. Figure 202 shows outlines of such gates. In Fig. 202a the jet is supported for some distance and the pressure at any depth is the same as for static conditions anywhere downstream from the vena contracta. The effective head is then h in the figure. If the jet is only partially supported, as in Fig. 202b, the pressure in the jet is diminished and the effective head is increased. With conditions as in Fig. 202c, the discharge is likely to be greater than for Fig. 202a or b. From the discussion of Fig. 168 in Art. 127 the

conclusion might be drawn that the supported length must be a minimum of about $12d$ if the true head is to be h in Fig. 202*a*. With conditions as in Fig. 202*d* the effective head is h, the difference in level of the surfaces. If a hydraulic jump is formed downstream from the vena contracta, then the head is increased by the distance s.

The computation of discharge from gates must be based upon experimental coefficients for conditions closely duplicating working conditions. The effects of sills, contractions and slightly rounded edges are difficult to predict and may result in erratic variations in coefficients.

General Problems

305. The jet from a 1-in. circular orifice in a vertical plane passes through a point 2 ft. below and 3 ft. horizontally from the vena contracta. The measured head on the orifice is 1.2 ft. and the measured discharge is 226 lb. of water in 2 min. Compute C_v and C_c.

306. A tank 4 ft. in diameter and 10 ft. high has a 2-in. circular orifice 9 ft. below the top. The water surface can be lowered 5 ft. from the top in 230 sec. by discharge through the orifice. Compute the coefficient of discharge.

307. A plate orifice 2 in. in diameter is placed in a 6-in. water line. What is the diameter of the jet? Find the drop in pressure when the discharge is 0.4 c.f.s. if C_v is 0.98. Compute the lost head.

308. Reservoir A, which has a water surface area of 2000 sq. ft., is connected to reservoir B by a 6-in. diameter standard short tube. The water surface in A is initially 6 ft. above the center of the tube and that in B is always 3 ft. above the center of the tube. What time is required to lower the surface in A 1 ft. by discharge through the tube?

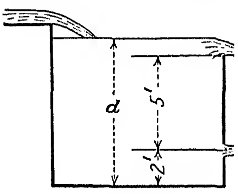


FIG. 203.

309. A short tube 2 in. in diameter is placed in the end of a 20-in. water line. What is the maximum discharge when the pressure in the pipe is (*a*) 16 lb. per sq. in. gage, (*b*) 20 lb. per sq. in. gage?

310. Water flows into the tank of Fig. 203 at a rate varying from 2 to 4 c.f.s. The tank has two outlets, a 6-in. circular orifice for which $C = 0.6$ and a 90-deg. V-notch weir 5 ft. above the center of the orifice. Compute maximum and minimum values of d and of Q for the orifice.

311. A 4-in. water turbine nozzle on a 12-in. pipe has coefficients $C_v = 0.97$ and $C_c = 0.94$. The pressure at the base of the nozzle is 100 lb. per sq. in. gage. What Q and what horsepower are delivered to the turbine?

312. A new 16-in. cast-iron water line with a well-rounded entrance leads 900 ft. from a reservoir to a turbine nozzle which is 300 ft. below the reservoir. The diameter of the jet at the vena contracta is 3 in. and $C_v = 0.97$. Find the head lost in the pipe, the head lost in the nozzle and the power

delivered to the wheel. What percentage of the total power is lost in delivering the water?

313. In Fig. 204 all the openings are 2 in. in diameter and the tubes are 6 in. long. Compute the effective head and the discharge for each opening. $C = 0.71$ for (a).

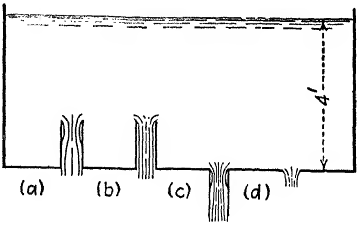


FIG. 204.

314. A large pipe carrying water at 50°C. and a pressure of 4 lb. per sq. in. gage leads to an orifice near the top of an airtight tank in which there is initially a perfect vacuum. The orifice is 2 in. in diameter and $C = 0.65$. Find the time required for 1500 gal. to flow into the tank.

315. A vertical cylindrical tank has an orifice for its outlet. When the water surface in the tank is 16 ft. above the orifice, the water can be lowered 12 ft. in 20 min., the pressure on the surface of water being atmospheric. What uniform air pressure must be applied to the surface if the same volume is to be discharged in 10 min.?

316. A vertical cylindrical tank has an orifice in its side at a point 2 ft. above the bottom. If the discharge reduces the depth of water in the tank from 18 to 12 ft. in 78 sec., what time is required to reduce the depth from 10 to 4 ft.?

317. A cylindrical tank 3 ft. in diameter and 10 ft. high contains 3 ft. of water, 4 ft. of oil (specific gravity, 0.8) and air at 16 lb. per sq. in. abs. Find the velocity and rate of discharge through a sharp-edged orifice 2 in. in diameter and 1 ft. above the bottom of the tank.

318. A tank car containing water has an orifice in the left end for which $C_v = 1$, $C_c = 1.0$ and $A = 0.025$ sq. ft. The water surface is 6.25 ft. above the opening. What is the discharge when the car is (a) at rest, (b) moving uniformly toward the left 8 ft. per sec.?

319. A jet of water from a 1-in. nozzle in the end of a 3-in. pipe line is discharged vertically upward. The velocity at the vena contracta is 50 ft. per sec., the diameter is 0.95 in. and $C_v = 0.96$. What is the diameter of the jet 20 ft. above the vena contracta? Find the pressure at the base of the nozzle if it is 2 ft. below the vena contracta.

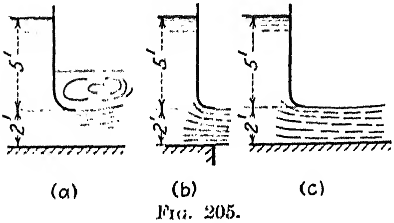


FIG. 205.

320. The coefficient of discharge in Fig. 205*a* is 0.9 and the total depth of water on the downstream side is 4 ft. Compute Q. Assuming the same coefficient, compute Q for Figs. 205*b* and *c*. The openings are 5 ft. wide.

CHAPTER XII

THE RESISTANCE OF IMMERSED AND FLOATING BODIES

156. Fluid Resistance.—When a solid body moves through a fluid, a resistance is produced which opposes this motion and energy must be expended in order that the motion may continue. Thus, when a submarine moves through the water or an airplane flies through the atmosphere, the propeller must supply a force acting in the direction of motion of sufficient magnitude to balance the resistance. A sailboat moves over the surface of an expanse of water if a wind current is directed against its sails so as to produce a propelling force. Its motion is opposed by a fluid resistance which consists of three separate parts, namely, the resistance due to the submerged portion of the hull moving through the water, the resistance due to the formation of waves on the surface of the water and the resistance caused by the motion through the air of the superstructure or portion of the ship above the water surface. It appears that, whenever there is a relative motion of a body with respect to a fluid, a resisting force results. A knowledge of the nature of this resistance is of great importance in engineering.

One of the earliest attempts to develop a rational theory of resistance was made by Newton. Although Newton's theory, using the momentum principle and based on the concept of a hypothetical fluid of discrete particles, does not agree at all with experimental data, the form of the expression is of considerable interest, even in connection with the modern idea of a fluid as a continuous medium. As a body moves through a fluid the particles of the latter are given momentum and the time rate of change of this quantity must, according to the momentum theorem, be equal to the force acting on the fluid. This force is equal and opposite to the resistance of the body.

Considering the changes in momentum in the direction of motion of the body, the quantity of fluid affected in unit time is

in general dependent on the velocity of the body and its size. The size may be conveniently considered as proportional to the area of the body projected on a plane normal to the direction of motion. If the fluid has a density ρ and the body is moving with a constant velocity V and has a projected area equal to A, then the mass of fluid affected by the body in 1 sec. is

$$m = k_1 \rho A V \tag{1}$$

where k_1 for the present is an unknown coefficient of proportionality. Now this mass of fluid which initially is at rest acquires an average velocity proportional to the velocity of the body, that is,

$$v = k_2 V \tag{2}$$

and consequently the momentum imparted to the fluid is

$$M = mv = k_1 k_2 \rho A V^2 \tag{3}$$

Since this last equation represents the momentum imparted to the fluid in unit time, it also represents the force acting on the body in the direction opposite to its motion. Following aeronautical parlance, this force or resistance is called drag and is denoted by the symbol D.

If the product of the two constants k_1 and k_2 in Eq. (3) is replaced by a single constant k_D, then the drag is

$$D = k_D \rho A V^2 \tag{4}$$

This is the Newtonian or so-called V-squared law of fluid resistance. It has the same form as the expression for the lift force acting on a vane or airfoil, developed in Chap. VII.

157. Drag Coefficients. The coefficient k_D of Eq. (4) is known as the drag coefficient. An inspection of Eq. (4) shows that it is a nondimensional quantity, and for this reason it is often spoken of as an absolute drag coefficient. The form of this equation for the drag is that commonly employed in England; k_D and the corresponding coefficient of lift, defined by the equation

$$L = k_L \rho A V^2 \tag{5}$$

are therefore called the English force coefficients. In the United States and in most other countries the expression for

drag is modified by multiplying and dividing by two and replacing the factor $2k_D$ by a new coefficient C_D so that

$$D = C_D \frac{\rho V^2}{2} A \tag{6}$$

It will be recalled that the expression $\rho V^2/2$ represents the dynamic pressure of a fluid stream, that is, it is the value of the rise in pressure at a stagnation point over that of the undisturbed stream. If this quantity is denoted by the symbol q, then the drag force is equal to

$$D = C_D q A \tag{7}$$

The drag coefficient from this equation is

$$C_D = \frac{D}{qA} \tag{8}$$

and C_D may therefore be interpreted as representing the ratio between the actual drag and the force that would be produced if the area A were acted on by a uniformly distributed pressure of magnitude q.

In some engineering work the equation for drag is written in terms of a coefficient which includes the density and is consequently no longer a pure number. This form, however, is gradually passing into disfavor since it does not bring out so clearly as do those given above the fact that the form of the resistance equation is essentially the same for all fluids. If a consistent set of units is used in either Eq. (4) or (6), the corresponding force coefficients will have the same numerical values whether the system used is metric or English. There exist certain hybrid coefficients which correspond to combinations of units obtained when density is expressed in slugs per cubic foot, velocity in miles per hour and area in square feet, but because of their clumsy form the use of these coefficients has little justification. In general the material presented here will be given in terms of the absolute coefficient C_D of Eqs. (6), (7) and (8).

It follows from either Eq. (6) or (8) that, if the density of a fluid, the velocity and area of a body moving through it and the magnitude of the resisting force are known, then the corresponding drag coefficient may be readily calculated. The same equations also hold in the case of the resistance produced when

the body is held stationary in an infinitely large current of fluid which, at great distance from the body, has a uniform velocity V. Some attempts have been made to determine the value of the resistance and corresponding coefficients by theoretical methods but these have been without success except in a very few special cases. In general the use of the mathematical theory of perfect fluids leads to a zero resistance, except in cases where surfaces of discontinuity or vortex trails exist in the wake of the body. This inability of the hydrodynamics of perfect fluids to give results which approximate those of experiments is known as the paradox of D'Alembert and is undoubtedly one of the principal reasons why this theory has had so little application in engineering work until recent years. Because of the complicated nature of problems in the determination of fluid resistance, quantitative information on this subject is obtained by experimental methods with a few notable exceptions.

Example.—A body having a projected area of 12.5 sq. ft. has a drag coefficient $C_D = 0.47$. It travels through air in a direction normal to the plane of its projected area. Determine the drag force and the power necessary to maintain the motion at velocities of 50 and 100 m.p.h.

Solution.—At 50 m.p.h. or 73.3 ft. per sec., assuming the air to be of standard density, the drag force is

$$D = C_D \frac{\rho V^2}{2} A = \frac{0.47 \times 0.002378 \times (73.3)^2 \times 12.5}{2} = 37.55 \text{ lb.}$$

The power is equal to the work done per second, that is,

$$P = DV = 37.55 \times 73.3 = 2755 \text{ ft. lb./sec.}$$

or

$$2755 \div 550 = 5 \text{ hp.}$$

The drag at 100 m.p.h. may be computed in the same manner or, noting that D is proportional to V^2,

$$D = 37.55(^{100}\!/\!_{50})^2 = 150.2 \text{ lb.}$$

If the power is written in the form

$$P = DV = \frac{C_D \rho V^3 A}{2}$$

it is seen that P is proportional to V^3. Then at 100 m.p.h.

$$P = 2755(^{100}\!/\!_{50})^3 = 22{,}000 \text{ ft. lb./sec.}$$
$$= 40 \text{ hp.}$$

Problem 321. A body having a drag coefficient $C_D = 0.25$ and a projected area of 5 sq. ft. moves through a fluid with a velocity of 60 m.p.h. Compute the drag force if the fluid is (*a*) water, (*b*) air.

322. A body moving through air is acted upon by a resistance of 250 lb. If the projected area is 7.5 sq. ft. and the velocity is 110 m.p.h., determine the values of the drag coefficients k_D and C_D.

323. A torpedo is launched with sufficient power so that, after it has reached a state of steady motion, its velocity in salt water ($w = 64.0$ lb. per cu. ft.) is 35 m.p.h. What speed would it attain in fresh water if (a) the resistance is the same, (b) the power is the same?

324. An automobile moving in still air has a drag coefficient $C_D = 0.45$. The projected area is 36 sq. ft. Determine the values of the resistance at speeds from 20 to 100 m.p.h. in intervals of 20 m.p.h. and plot the results as functions of V and V^2. What type of curves are obtained?

158. The Effects of Viscosity.—The momentum theory of resistance developed in the preceding articles does not consider the effect of viscosity. It seems natural to expect, however, that this property of fluids should be one of the chief causes of such resistance. The manner in which viscosity affects the drag may be determined by the methods of dimensional analysis. If it is assumed that the drag of a body is dependent on the velocity, the projected area and the density and viscosity of the fluid, then the equation relating these quantities may be written in the form

$$D = k_D \rho^a A^b V^c \mu^d \tag{9}$$

where k_D as before is an absolute or nondimensional coefficient and where the exponents a, b, c and d are to be determined. The condition of dimensional homogeneity of Eq. (9) leads to the relationship

$$M\frac{L}{T^2} \approx \left(\frac{M}{L^3}\right)^a L^{2b}\left(\frac{L}{T}\right)^c\left(\frac{M}{LT}\right)^d$$

where M, L and T are the fundamental dimensions of mass, length and time, respectively. On equating the exponents of each of these dimensions independently, three simultaneous linear equations are obtained, from which three of the unknown exponents may be calculated in terms of the fourth. These three equations are

$$\begin{aligned} 1 &= a + d \\ 1 &= -3a + 2b + c - d \\ -2 &= -c - d \end{aligned}$$

and the values of a, b and c obtained therefrom in terms of d are

$$a = 1 - d$$
$$b = 1 - \frac{d}{2}$$
$$c = 2 - d$$

so that the resistance equation becomes

$$D = k_D \rho A V^2 \left(\frac{\mu}{\rho A^{1/2} V} \right)^d$$

The term $A^{1/2}$ in the parentheses may be replaced by a linear factor l, which is some characteristic dimension of the body, by simply changing the value of the drag coefficient. The result is then

$$D = k_D' \rho A V^2 \left(\frac{\mu}{\rho V l} \right)^d$$

Multiplication and division by the factor 2 make it possible to replace the coefficient k_D' by the coefficient C_D', thus making the expression for the drag directly comparable with Eq. (6). This modified form of the expression for the drag is

$$D = C_D' \frac{\rho V^2}{2} A \left(\frac{\mu}{\rho V l} \right)^d$$

The expression $\rho V l / \mu$ will be recognized as N_R, the Reynolds' number of the flow, and the resistance equation in terms of N_R is then

$$D = C_D' \frac{\rho V^2}{2} \frac{A}{N_R{}^d} \tag{10}$$

It thus appears that this number is of considerable importance in determining the nature of the resistance of a body moving through a viscous fluid. A comparison of Eqs. (6) and (10) shows that when viscosity is taken into account the former equation is correct only when its value of C_D is considered as a function of Reynolds' number. The coefficients in these equations are therefore not identical but

$$C_D \text{ (Eq. 6)} = \frac{C_D' \text{ (Eq. 10)}}{N_R{}^d}$$

The linear dimension l which appears in the above expression for Reynolds' number is to a certain extent an arbitrary quantity,

but in most cases it is taken as the principal dimension of the body measured as nearly as possible in the direction of motion. In some problems it is either impossible or inconvenient to employ such a dimension, and some other length which is more characteristic of the shape of the body may be selected. For instance, in the case of a thin circular disk moving in a direction normal to its surface, the dimension in the direction of motion would be the thickness of the disk, which is of little significance in determining the value of the resistance and is usually replaced by the diameter. In the case of a sphere the resistance is the same for all directions of motion and the characteristic length is obviously the diameter. In Eq. (31) of Art. 93 Stokes' law for the resistance of a sphere in "creeping motion" was given in the form

$$D = \frac{3\pi\rho V^2 d^2}{N_R}$$

where $N_R = \rho V d/\mu$. In this case the projected area is $A = \pi d^2/4$ so that

$$D = \frac{12\rho V^2 A}{N_R}$$

and the drag coefficient as defined by Eq. (6) is

$$C_D = \frac{D}{\dfrac{\rho V^2}{2} A} = \frac{24}{N_R} \tag{11}$$

which shows that C_D is a function of the Reynolds' number.

In some cases the nature of the relationship between C_D and N_R may be determined by theoretical methods but in the majority of problems it is necessary to resort to experimental means because of the difficulties in setting up and solving the mathematical equations that accurately represent these fluid motions.

More detailed studies of resistance show that the Reynolds' number of any flow may be considered as an index of the relative importance of the inertia and viscous forces involved in the fluid motion. In Chap. XV it will be shown that N_R is actually proportional to the ratio between these forces, that is,

$$N_R \propto \frac{\text{inertia force}}{\text{viscous force}}$$

Although a proof of this statement will not be given at present, this relation may be used to considerable advantage in explaining the variation of drag with Reynolds' number. When the Reynolds' number is small the viscous forces are large as compared with those due to the inertia of the fluid particles, while when N_R is large the opposite is true. In the first case the motion is determined entirely by the viscosity of the fluid and is independent of its inertia and therefore of its density. Stokes' solution for the drag of a sphere as given above and by Eq. (20) of Art. 87 is based on this type of flow. However, it is found that for bodies of any shape the drag is given by a similar equation of the form

$$D = kVl\mu$$

so that the drag coefficient is

$$C_D = \frac{D}{\frac{\rho V^2}{2}A} = \frac{2kVl\mu}{\rho V^2 A} = \frac{2k}{\frac{\rho VA}{\mu l}}$$

or, putting $A = k'l^2$,

$$C_D = \frac{2k}{\rho Vlk'} = \frac{K}{N_R}$$

The coefficients k and K depend on the shape of the body.

For large values of Reynolds' number the viscosity has no appreciable effect on the flow so that the drag is independent of N_R. The exponent d in Eq. (10) is then equal to zero and the drag coefficient C_D of Eq. (6) is a constant for a body of a given shape.

It thus appears that when N_R is small the drag force on a body is proportional to the first power of the velocity and C_D is inversely proportional to N_R. When the Reynolds' number is very large, the drag is proportional to the square of the velocity and C_D is independent of N_R. For intermediate values of the Reynolds' number it may be expected that the drag will depend on some power of the velocity between 1 and 2 while C_D will depend on the Reynolds' number to a power between -1 and 0. In the articles which follow a survey will be given of the most important theoretical and experimental results that have been obtained for bodies of various shapes.

Problem 325. A body having a length of 5 ft. moves through a fluid with a velocity of 75 m.p.h. What is its Reynolds' number for air and water at a temperature of 59°F.?

326. A sphere 0.01 ft. in diameter moves through oil having a kinematic viscosity of 0.002 ft.^{2} per sec. at a velocity of 0.075 ft. per sec. Determine the drag coefficient and the drag force if the specific gravity is 0.82.

327. A body 3 ft. long moves through air and its Reynolds' number is 3×10^6. At what velocity must it move through water at 59°F. in order to have the same drag coefficient?

159. The Boundary-layer Theory.—The majority of engineering problems which involve questions of fluid resistance are concerned with the motion of bodies through air or water and, since these are both fluids of relatively small viscosity, it might be assumed that the drag of objects moving through them is practically independent of Reynolds' number. This hypothesis implies, as will be shown in Chap. XV, that the inertia forces acting on the particles of fluid are so much larger than the viscous forces that the latter may be neglected. Such an assumption formed the basis of the early attempts to apply the mathematical theory of nonviscous fluids but, as has already been mentioned, these methods were not very successful, leading to a zero value for the resistance to the steady motion of a body except in cases where account was taken of the formation of a wake.

The principal reason for the inability of perfect-fluid theory even to approximate actual values of resistance is the fact that the hypothesis of zero slip at the boundary of the solid has been abandoned. It will be recalled that the shearing stress in a viscous fluid is proportional to the velocity gradient normal to the direction of the flow as well as to the viscosity so that, even though the latter value is extremely small, stresses of appreciable magnitude will exist if the velocity gradient is large. Such conditions may exist near the surface of a solid body in a stream of fluid where the velocity changes rapidly from zero at the surface of the body to a value that may be quite large at a small distance from the body. These conditions were first taken into consideration by Prandtl[1] in formulating his boundary-layer theory which was first published in 1904.

[1] PRANDTL, L., On the Motion of Fluids with Very Little Viscosity, *NACA Tech. Memo.* 452. Translated from "Vier Abhandlungen zur Hydrodynamik und Aerodynamik," Göttingen, 1927.

The basis of Prandtl's theory is that the fluid surrounding the body may be divided into two portions: (1) a thin layer close to the surface of the body in which the velocity gradient is large enough to produce viscous forces of an appreciable magnitude; (2) the remaining portion of the fluid outside this boundary layer, in which the viscous forces may be neglected in comparison with the inertia forces, or, in other words, in which the Reynolds' number may be assumed to be infinitely large. Under the assumption that the thickness of the boundary layer is small and that the radius of curvature of the surface of the body is large as compared to this thickness, Prandtl was able to simplify the differential equations which represent the motion of a viscous fluid. This was an extremely important advance in the development of fluid mechanics because of the fact that previously only such special problems as Stokes' solution for the sphere and the Hagen-Poiseuille law for pipe flow had been obtained from the more general equations first set up by Navier and Stokes as early as 1827.[1]

The mathematics of the boundary-layer equations is too advanced for this text; only the physical basis of the theory and some of the results will be discussed here. One of the important facts deduced from the theory of the boundary layer is that pressures are transmitted without change through the layer in directions normal to the bounding surface. Because of this fact the velocity at the outer limit of the boundary layer may be computed from an experimentally determined pressure distribution on the body by using Bernoulli's theorem. In some cases the methods of the hydrodynamics of perfect fluids may be employed for this purpose, provided the contour of the body and its surrounding boundary layer are amenable to mathematical treatment.

160. Laminar and Turbulent Boundary Layers.—In the study of the flow of fluids through pipes it was pointed out that at the critical Reynolds' number the flow begins to change from a laminar to a turbulent character, and the expressions for the

[1] The general equations for the motion of viscous fluids were first published by Navier in 1827, by Poisson in 1831, on a different basis by St. Venant in 1843 and by Stokes in 1845. See L. Prandtl and O. G. Tietjens, "Fundamentals of Hydro- and Aeromechanics," p. 259, McGraw-Hill Book Company, Inc., New York, 1934.

drop in pressure along the pipe are decidedly different in form in the two cases. When a fluid of small viscosity moves past any solid surface, the flow in the boundary layer behaves in a similar manner. The characteristic length which appears in the Reynolds' number here is taken as the length of the surface measured in the direction of motion. For low values of this Reynolds' number the velocity distribution in the boundary layer corresponds to laminar flow. As the Reynolds' number is increased the boundary layer begins to show the characteristics of turbulence, this condition starting at the rear or tail end of the surface. For further increases in Reynolds' number the point of transition from laminar to turbulent flow moves forward along the surface until eventually, at a high value of N_R, the entire boundary layer is turbulent. In the transition régime the change from laminar to turbulent flow does not occur at a definite point on the surface, although in the theoretical work on these problems the assumption is made that such is the case. Actually the change takes place more or less gradually over an appreciable length of the surface in a manner analogous to the corresponding change in the flow through a pipe.

The variation of the velocity with distance from the surface is different for laminar and for turbulent flow. In the exact solution of these problems the velocity in the boundary layer approaches asymptotically the velocity determined by the pressure distribution on the body so that the thickness of the boundary layer does not have a definite value. Approximate expressions for the boundary-layer thickness may be obtained by various methods, one of which is to define the thickness as that distance from the surface at which the velocity differs from that of the flow outside the layer by an arbitrary amount, say 1 per cent of the latter velocity. In general the determination of the shear stress and resistance requires a knowledge of the velocity variation along the surface at the outer limit of the boundary layer.

161. Transverse Velocity Distribution in Boundary Layers.—The first solution of Prandtl's boundary-layer equations was given by Blasius[1] for the case of the flow of an infinitely large stream of fluid past a thin flat plate parallel to the direction of

[1] BLASIUS, H., Grenzschichten in Flüssigkeiten mit kleiner Reibung, *Z. Math. Physik*, vol. 56, 1908.

motion. This problem is extremely complicated in its mathematical details, but a considerable simplification in the method of treatment was introduced by von Kármán[1] by means of a consideration of the changes in momentum that occur in an element of the boundary layer. The application of this method requires an assumption as to the variation of velocity in the boundary layer with the distance normal to the bounding surface. For a laminar boundary layer fairly satisfactory results can be obtained by assuming a parabolic velocity distribution. If x and y are coordinates measured along and normal to the surface as shown

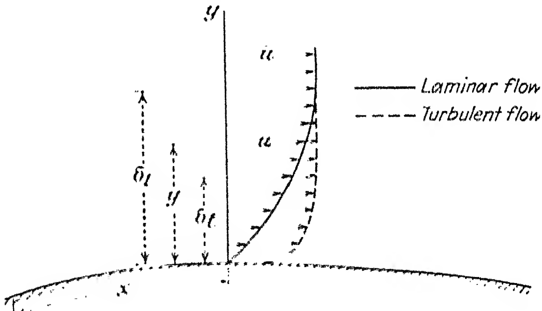


FIG. 206.—Velocity distribution in laminar and turbulent boundary layers.

in Fig. 206, then, for any value of x, the velocity parallel to the surface may be considered as a function of y of the form

$$by \qquad cy^2 \tag{12}$$

The terms a, b and c are constants determined by the three conditions:

1. No slip at the boundary and therefore $u = 0$ when $y = 0$.
2. At the outer edge of the boundary layer the velocity is $\bar{u}$; therefore $u = \bar{u}$ when $y = \delta$, the boundary-layer thickness.
3. At the outer edge of the boundary layer the tangent to the velocity distribution curve is parallel to the y-axis so that

$$\frac{\partial u}{\partial y} \qquad \text{when} \qquad y = \delta.$$

When these conditions are satisfied, Eq. (12) becomes

[1] VON KÁRMÁN, TH., Über laminare und turbulente Reibung, *Z. angew. Math. Mech.*, vol. I, no. 1. 1921.

$$u = \bar{u}\frac{y}{\delta}\left(2 - \frac{y}{\delta}\right) \tag{13}$$

The distribution represented by this equation is shown by the solid curve in Fig. 206.

For the case of a turbulent boundary layer von Kármán showed that the velocity in the boundary layer varies as the $\frac{1}{7}$ power of the distance from the wall, the exact relation being

$$u = \bar{u}\left(\frac{y}{\delta}\right)^{\frac{1}{7}} \tag{14}$$

This distribution is represented by the dotted line in Fig. 206. This expression was obtained from an analysis of Blasius' law for the pressure drop in pipes with turbulent flow. In the case of the pipe the entire flow may be regarded as a boundary layer where the thickness δ corresponds to the radius r. If δ is replaced by r and $\bar{u}$ is replaced by the velocity at the center of the pipe, Eq. (14) becomes the same as Eq. (36) of Art. 107.

Problem 328. The thickness of a boundary layer is 2 in. at a certain point on a body when the velocity at the outer edge of the layer is 10 ft. per sec. Draw curves showing the velocity distribution for laminar and turbulent flow by plotting values of velocity at intervals of 0.5 in.

329. The boundary-layer thickness at a certain point on a body is 1 in. for laminar flow and $\frac{1}{2}$ in. for turbulent flow, the velocity at the outer edge being 25 ft. per sec. in both cases. Draw curves showing the velocity distributions under these conditions by computing the velocities for $y = 0.25\delta$, 0.50δ and 0.75δ.

330. Calculate the shearing stresses at intervals of 0.5 in. in a laminar boundary layer 2.5 in. thick when the velocity at the outer edge is 15 ft. per sec. and the fluid is water at 15°C.

162. Separation of Boundary Layers. The calculation of resistance is a perfectly straightforward process if the boundary layer remains in contact with the body over the entire surface. The thin flat plate set parallel to the direction of its motion satisfies this condition and, as will be shown later, the theory is in good agreement with experimental results. The boundary layer also remains in contact with the surface of other bodies such as cylinders (struts and airfoils) and bodies of revolution (ellipsoids, airship hulls, etc.) provided neither the inclination of the body to the direction of motion nor the ratio of its thickness to length is too great. When the inclination or this ratio exceeds

certain limits, the boundary layer on the rear portion of the body may detach itself from the surface and a surface of discontinuity forms immediately in the rear. Because of its inherent instability this detached layer usually rolls up into vortices which are left behind the body and form what is known as the wake.

The process of separation of the boundary layer and the formation of a wake produces a region of reduced pressure over the rear of the body, and as a consequence the resistance is materially increased. The total drag of the body is then made up of the direct effect of viscosity in the production of shearing stresses along those parts of the surface where the layer has not separated and the indirect effect of viscosity which results in the formation of a wake.

163. The Mechanism of Separation.—The mechanism involved in boundary-layer separation can be explained by considering the case of a circular cylinder immersed in a stream which moves in the direction perpendicular to its axis. Suppose the cylinder to be of sufficient length so that the flow may be regarded as two-dimensional. The classical hydrodynamics of perfect fluids shows that the streamlines of the motion are defined by the equation

$$\Psi = -V\left(r - \frac{a^2}{r}\right) \sin \theta \tag{15}$$

where r and θ are the polar coordinates of any point in a plane cross section of the flow referred to the center of the circular cross section of radius a. The velocity of the stream at a great distance from the cylinder is V and Ψ has a constant numerical value for each individual streamline. The flow is in the direction $\theta = \pi$ at a great distance from the cylinder. The nature of these streamlines is illustrated by Fig. 207. The velocity at any point M in the fluid has the radial and tangential components

$$V_r = -V\left(1 - \frac{a^2}{r^2}\right) \cos \theta \tag{16a}$$

$$V_\theta = V\left(1 + \frac{a^2}{r^2}\right) \sin \theta \tag{16b}$$

On the boundary of the cylinder $r = a$ and $V_r = 0$; the velocity is purely tangential and has the magnitude

$$V_c = 2V \sin \theta \tag{17}$$

From the symmetry of the flow about the vertical axis, the pressures at corresponding points on the front and rear of the cylinder are equal and there is no drag. If the flow of a slightly viscous fluid such as air or water resembled that of Fig. 207, then Eqs. (16*a*) and (16*b*) could be used to determine the velocity at the outer edge of the boundary layer and the drag could be computed. However, the flow shown in Fig. 207 does not completely resemble that which actually exists for usual values of the Reynolds' number because separation takes place. This separation is the combined effect of energy losses due to viscosity and of the pressure distribution over the surface of the cylinder. The

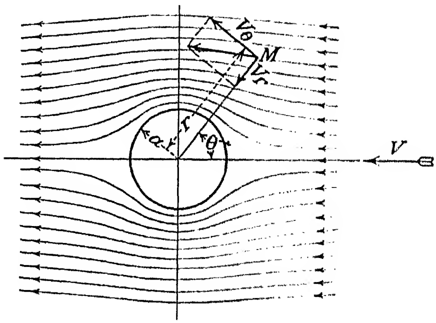


FIG. 207.—Motion of a perfect fluid past a long circular cylinder.

theoretical pressure distribution corresponding to the perfect fluid motion of Eq. (15) was developed in Art. 68 and was shown in Fig. 107. The ratio of the gage pressure at any point on the cylinder to the dynamic pressure of the stream as given by Eq. (3) of Art. 68 is

$$\frac{p - p_0}{\rho V_0^2/2} \quad 1 - 4 \sin^2 \tag{18}$$

The values of this ratio are plotted in Fig. 208 in exactly the same manner used in obtaining Fig. 107.

In the flow of a perfect fluid a particle near the surface of the cylinder experiences changes in its kinetic energy as it moves around the circle, but these changes are exactly balanced by the changes in pressure so that, when a particle moves from the forward end of the horizontal diameter to the downstream end, its total energy remains unchanged. It will be noted in Fig. 208 that between points C and E on the surface the pressures are

increasing toward E so that there is a pressure gradient opposing the motion of the particle in this region. Thus a particle moving along the surface must have acquired enough additional kinetic energy in the first part of its journey between A and C to enable it to continue around the circle to point E. In the motion of a perfect fluid this condition is just satisfied, but when the fluid is viscous there is a continual decrease in the kinetic energy in the direction of motion so that somewhere between points C and E a point is reached where the particle can no longer continue its motion toward point E against the pressure but starts to reverse its direction. It immediately collides with other particles still moving rearward from C and the resulting disturbance causes the boundary layer to separate from the surface, thereby forming the wake behind the body. The resulting flow after these changes have been stabilized is shown in Fig. 209.

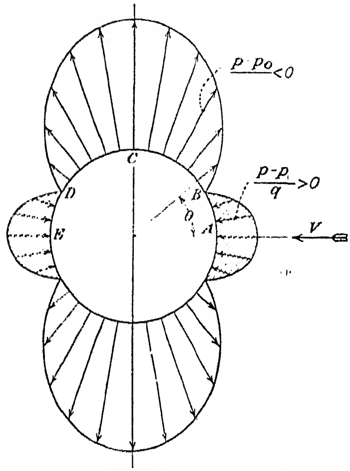


FIG. 208.—Theoretical pressure distribution on a circular cylinder.

The pressure distribution is no longer that of Fig. 208 although the changes which result from separation are confined principally to the rear portion of the cylinder, as is shown by the experimentally determined values which are plotted in Fig. 210.

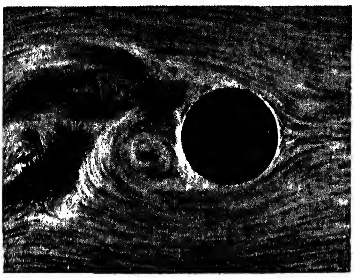


FIG. 209.—Actual flow past a long circular cylinder.[1]

As a result of this discussion of separation it is seen that there are in general three distinct regions in the fluid through which a body is moving, instead of two as previously mentioned. In addition to the boundary layer and the fluid outside it, one must consider the possible existence of a wake behind the body in which there is an eddying motion.

[1] PRANDTL and TIETJENS, *op. cit.*, p. 279, Fig. 5.

The resistance of a body may now be considered as made up of two parts: (1) the direct effect of viscosity producing shearing stresses, the resultant of which is called skin-friction drag, and (2) the drag caused by the formation of the wake, the latter being commonly known as eddy-making resistance. The shape of the body determines the character of the pressure distribution on its surface, and it is possible to construct bodies on which the changes in pressure are very gradual so that, if any separation occurs at all, it is very close to the rear and the wake produced is extremely narrow. The eddy-making resistance in such cases is either negligible or only a very small part of the total drag, most of the drag being due to skin friction because the boundary layer now covers almost the entire surface. Bodies for which the eddy-making drag is extremely small are known as streamlined shapes because the flow around them is of approximately the same character as the streamline flow that would be obtained with a hypothetical nonviscous fluid. A "perfectly streamlined body" might be defined as one for which the eddy-making drag is zero.

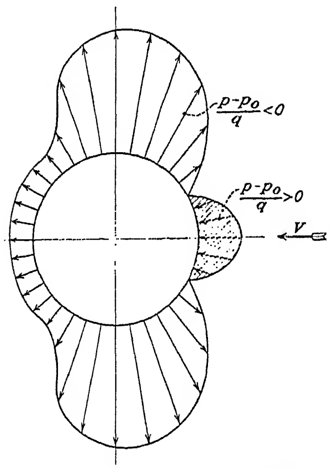


FIG. 210.—Actual pressure distribution on a circular cylinder.

Problem 331. A stream of air having a velocity of 50 ft. per sec. moves past a circular cylinder in a direction perpendicular to its axis. Assuming the air to be nonviscous, determine the values of the gage pressure, velocity and kinetic and total energies per unit volume for points on the surface of the cylinder. Plot the results as functions of θ, using a rectangular system of axes and taking θ in increments of 10 deg.

164. Effects of Laminar and Turbulent Flow on Separation.—In the discussion of separation just completed, the question of whether the boundary layer on the body ahead of the separation point was laminar or turbulent was not considered. It has been shown experimentally that the nature of the boundary-layer flow not only affects the value of the skin-friction drag but that it also plays an important part in determining the

magnitude of the eddy making drag. That this is the case may be shown by a detailed study of the flow in the boundary layer in the neighborhood of the separation point. Figure 211 shows a

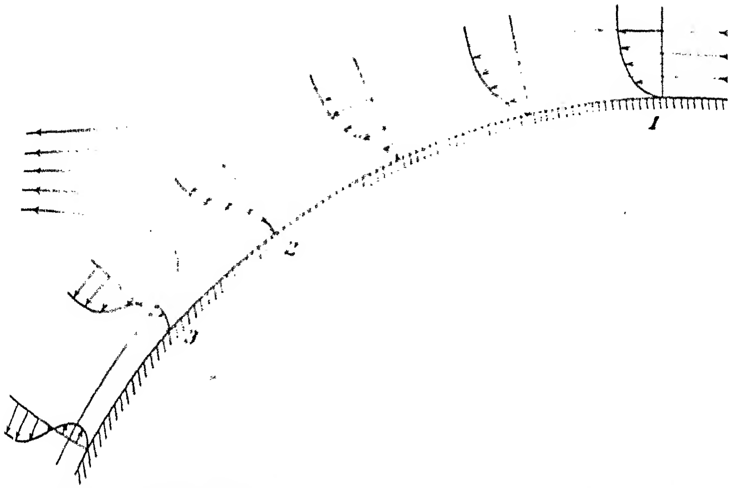


FIG. 211a. Boundary layer velocity distribution near a separation point.

FIG. 211b. Boundary layer flow near separation point.

magnified sketch of the boundary layer between the points C and E of Fig. 208 as well as a photograph of an actual flow[1] showing a particular stage in the development of the eddying

[1] PRANDTL and TIETJENS, *op. cit.*, p. 290, Fig. 29.

flow which results from separation. The velocity-distribution curves for several points in this region are also drawn, the one on the extreme right at point 1 corresponding to a profile of the type defined by either Eq. (13) for laminar flow or Eq. (14) for turbulent flow. This profile is one which has not yet been greatly modified by the loss in energy due to viscosity but, as the motion is followed toward the rear part of the body, the change in the boundary-layer velocity profiles becomes apparent. The particles nearest to the surface are, of course, retarded the most so that there is a general steepening of the curves corresponding to a decrease in the velocity gradients. The separation point is defined as the point where the tangent to the velocity-distribution curve at the surface becomes normal to the surface, and in Fig. 211*a* this occurs at point 2. At point 3 the profile shows the reversal of flow which leads to separation, while the general character of the flow as a whole is indicated by the streamlines that have been sketched in.

Experiments have shown conclusively that, when the boundary layer is turbulent, separation does not occur quite so soon as in the case of a laminar boundary layer. This conclusion may also be reached by an examination of the laminar and turbulent velocity profiles of Fig. 206. The laminar distribution curve has a tangent at the boundary which makes an angle $\frac{\pi}{2} - \tan^{-1}\frac{2\bar{u}}{\delta}$ with the tangent to the surface, whereas for turbulent flow this angle is zero. Thus, to make the tangent normal to the surface, more energy must be dissipated for the turbulent flow than for the laminar one, with the result that the separation point is farther toward the rear of the cylinder in the former case. It may be concluded that, because of the smaller wake, the eddy-making resistance with a turbulent boundary layer on the forward portion of the cylinder will be appreciably less than when the boundary layer is laminar.

165. Skin-friction Drag of a Thin Plate.—The thin flat plate placed parallel to the motion of a uniform stream in which it is immersed is an example of fluid motion for which there is no separation and for which the resistance is therefore due entirely to skin friction. As has already been mentioned, a theoretical solution to this problem for laminar flow has been obtained by

Blasius. Pohlhausen[1] has applied von Kármán's momentum theorem in a manner which gives approximate results agreeing very well with the exact solution. Von Kármán and Prandtl have worked out independently the case of flow with a turbulent boundary layer. In all these solutions the assumption is made that the velocity and pressure at the outer edge of the boundary layer do not vary in the direction of motion and that the former value is equal to the velocity of the stream. Under these conditions the momentum theorem may be applied[2] by con-

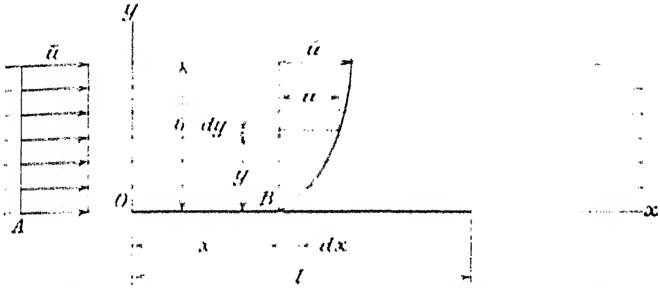


FIG. 212. Boundary layer velocity distribution for a flat plate.

sidering the transverse velocity distribution upstream from the plate and at any point on the plate. Upstream from the plate in Fig. 212, the velocity is uniform. A set of rectangular axes, x and y, is chosen so that the x-axis coincides with the plate and the origin is at its forward edge. At any point x on the plate the velocity distribution is assumed to be of the form

$$u \; f| \tag{19}$$

The boundary-layer thickness δ increases in the direction of motion and is therefore a function of x. Since the velocity $\bar{u}$ at the outer edge of the layer is assumed constant, the ratio of the velocity u within the layer to $\bar{u}$ will be the same for a given value of y/δ for all values of x. The velocity profiles are therefore of the same general shape. The arrangement of

[1] POHLHAUSEN, K., Zur näherungsweisen Integration der Differentialgleichung der laminaren Grenzschicht, *Z. angew. Math. Mech.*, vol. I, no. 1, 1921.

[2] VON KÁRMÁN, TH., Turbulence and Skin Friction, *J. Aero. Sci.*, vol. 1, no. 1, 1934.

axes and the velocity profile for point A upstream and for point B on the plate are shown in Fig. 212.

Considering the plate to have width b, the mass of fluid passing through an element of thickness dy of the boundary layer at B is $\rho ub\,dy$ in unit time and its momentum is $\rho u^2 b\,dy$. When this mass of fluid passed the cross section at A, it had a velocity $\bar{u}$ and its momentum was initially $\rho\bar{u}ub\,dy$. The net change in momentum for this element of fluid is therefore $\rho ub(\bar{u} - u)dy$ and for the entire boundary layer at B it is

$$\rho b \int_0^\delta u(\bar{u} - u)dy$$

This change in momentum is the force due to skin friction acting on the plate from the forward edge up to the point B. This force for one side of the plate only has the value

$$D_{fx} = \rho b \int_0^\delta u(\bar{u} - u)dy \tag{20}$$

For an elementary strip of area of the surface of width dx, the frictional force is

$$dD_{fx} = b\tau_0\,dx$$

where τ_0 is the shearing stress at the plate at this point. The value of τ_0 may therefore be written in the form

$$\tau_0 = \frac{1}{b}\frac{dD_{fx}}{dx} = \rho\,\frac{d}{dx}\int_0^\delta u(\bar{u} - u)dy \tag{21}$$

If the value of u from Eq. (19) is introduced in Eq. (20), the drag becomes

$$D_{fx} = \rho b\bar{u}^2 \int_0^\delta f\left(\frac{y}{\delta}\right)\left[1 - f\left(\frac{y}{\delta}\right)\right]dy$$

Replacing the relative ordinate y/δ by η and putting $dy = \delta\,d\eta$, this expression becomes

$$D_{fx} = \rho b\bar{u}^2\delta \int_0^1 f(\eta)[1 - f(\eta)]d\eta$$

Now letting

$$\alpha = \int_0^1 f(\eta)[1 - f(\eta)]d\eta \tag{22}$$

the drag is

$$D_{fx} = \rho b\bar{u}^2\delta\alpha \tag{23}$$

Substituting this value of D_{fx} in Eq. (21) and noting that α is independent of x, the value of the shearing stress becomes

$$\tau_0 = \rho \bar{u}^2 \alpha \frac{d\delta}{dx} \tag{24}$$

This stress may also be expressed in the alternative form

$$\tau_0 = \mu \left(\frac{du}{dy} \right)_{y=0} = \frac{\mu}{\delta} \left(\frac{du}{d\eta} \right)_{\eta=0}$$

or, putting $u = \bar{u}\, f(\eta)$ and

$$\beta = \left[\frac{df(\eta)}{d\eta} \right]_{\eta=0} \tag{25}$$

$$\tau_0 = \frac{\mu \bar{u} \beta}{\delta} \tag{26}$$

Equating expressions (24) and (26) for the shearing stress leads to the differential equation

$$\delta \frac{d\delta}{dx} = \frac{\mu\beta}{\rho \bar{u} \alpha}$$

This can be integrated and, since the boundary-layer thickness is zero at the leading edge, the constant of integration is zero. The value of the boundary-layer thickness is then

$$\delta = \sqrt{\frac{2\beta}{\alpha} \frac{\mu x}{\rho \bar{u}}} \tag{27}$$

If this value of δ is introduced in Eq. (26), the shearing stress is

(28)

The total drag may then be computed by integrating the elementary shear force over the surface so that

$$D_{fx} = b \int_0^x \tau_0 \, dx = b\sqrt{2\alpha\beta\rho\mu\bar{u}^3 x} \tag{29}$$

For the entire surface of one side of the plate the length is $x = l$ so that the total drag is

$$D_f = b\sqrt{2\alpha\beta\rho\mu\bar{u}^3 l} \tag{30}$$

It is desired to express this drag in terms of a drag coefficient based on the dynamic pressure and some characteristic area, as was done in Eq. (6). In the case of the plate this area is taken as that of the exposed surface so that for one side of the plate the area is bl and the drag coefficient is

$$C_f = \frac{2D_f}{\rho \bar{u}^2 bl} = 2\sqrt{\frac{2\alpha\beta\mu}{\rho \bar{u} l}} = 2\sqrt{\frac{2\alpha\beta}{N_R}} \tag{31}$$

where $N_R = \rho \bar{u} l/\mu$, the Reynolds' number.

166. Skin Friction for Laminar Boundary Layer.—When the general theory outlined above is applied to the case of the laminar boundary layer, the assumption of a parabolic velocity distribution as defined by Eq. (13) is introduced. The solution is completed by determining $f(\eta)$ from Eq. (13) and the values of α and β from Eqs. (22) and (25). In this case these values are

$$\alpha = 2/15 \qquad \beta = 2$$

When these values are introduced in Eqs. (27), (28) and (31), the resulting expressions for the boundary-layer thickness, the shearing stress and the drag coefficient are as follows:

$$\delta = 5.48\sqrt{\frac{\mu x}{\rho \bar{u}}} = 5.48\sqrt{\frac{xl}{N_R}} \tag{32}$$

$$\tau_0 = 0.365\sqrt{\frac{\rho\mu\bar{u}^3}{x}} = 0.730\,\frac{\rho \bar{u}^2}{2}\sqrt{\frac{l}{xN_R}} \tag{33}$$

$$C_f = \frac{1.460}{\sqrt{N_R}} \tag{34a}$$

The exact solution of Blasius gives an expression for the drag coefficient of the same form as Eq. (34*a*) with the exception that the numerical coefficient is 1.327. The exact expression

$$C_f = \frac{1.327}{\sqrt{N_R}} \tag{34b}$$

agrees somewhat better with experimental data and will be employed in the remainder of this work.

167. Skin Friction for Turbulent Boundary Layer.—When the boundary layer is turbulent, the use of the $1/7$-power-law velocity distribution of Eq. (14) would seem at first glance to imply that

the shearing stress at the wall is infinitely large if the relation $\tau_0 = \mu(du/dy)_{y=0}$ is employed. This difficulty is avoided by making use of the close analogy that exists between boundary-layer flow and the flow in pipes. In Art. 106, Eq. (30), it was shown that the shearing stress at the wall of a pipe in which the flow is turbulent is

$$\tau = \frac{0.0395\rho V^2}{\sqrt[4]{N_R}}$$

where V is the average velocity and $N_R = \rho V d/\mu$. This expression was obtained from Blasius' empirical law for pressure drop which forms the basis for the $\frac{1}{7}$-power law for velocity distribution.

The maximum velocity at the center of the pipe is $u_{\text{max.}} = 1.235V$ and, if this relation is introduced in the above expression and at the same time $d = 2r$ is substituted, the shearing stress becomes

$$\tau = \frac{\dfrac{0.0395}{(1.235)^2}\rho u_{\text{max.}}{}^2}{\sqrt[4]{\dfrac{2}{1.235}\dfrac{\rho u_{\text{max.}} r}{\mu}}} = \frac{0.023\rho u_{\text{max.}}{}^2}{}$$

In developing the $\frac{1}{7}$-power law it was assumed that τ is independent of the radius of the pipe for a given Reynolds' number. The boundary layer on a flat plate may be regarded as the same as that in the pipe if it is imagined that the pipe is cut longitudinally and its surface unrolled so as to form a plane. The radius of the pipe then corresponds to the boundary-layer thickness δ, and the velocity at the center is equivalent to $\bar{u}$, the velocity at the outer edge of the boundary layer on the plate. The shearing stress at the surface of the plate is then

$$\tau_0 = \frac{0.023\rho\bar{u}^2}{\sqrt[4]{\text{[illegible]}}} \tag{35}$$

When formulas (35) and (24) for the value of τ_0 are equated, a differential equation for δ is obtained which may be written in the form

$$\delta^{1/4}\frac{d\delta}{dx} = \frac{0.023}{\alpha}\left(\frac{\mu}{\rho\bar{u}}\right)^{1/4}$$

so that after integration δ is

$$\delta = \left(\frac{0.0288}{\alpha}\right)^{4/5}\left(\frac{\mu}{\rho\bar{u}x}\right)^{1/5} x \tag{36}$$

The value of u from Eq. (14) is now introduced so as to determine α, which, as in the case of laminar flow, is defined by Eq. (22). The result is

$$\alpha = 7/72$$

When this is substituted in Eq. (36), the boundary-layer thickness becomes

$$= 0.377\left(\frac{\mu}{\rho\bar{u}x}\right)^{1/5} x \tag{37}$$

This expression for δ may now be introduced in Eq. (35) for shearing stress, which finally is

$$\tau_0 = 0.0587\,\frac{\rho\bar{u}^2}{2}\left(\frac{\mu}{\rho\bar{u}x}\right)^{1/5} \tag{38}$$

The drag of one side of the entire plate of length l is found by evaluating the integral in the expression

$$D_f = b\int_0^l \tau_0\,dx$$

which leads to the result

$$D_f = 0.073bl\,\frac{\rho\bar{u}^2}{2}\left(\frac{\mu}{\rho\bar{u}l}\right)^{1/5} \tag{39}$$

The drag coefficient is then equal to

$$C_f = \frac{0.073}{(N_R)^{1/5}} \tag{40a}$$

But in order to obtain a slightly better agreement with experimental data, the numerical coefficient is changed to 0.074 so that

$$C_f = \frac{0.074}{(N_R)^{1/5}} \tag{40b}$$

From a physical standpoint there might be some objection to the use of the $1/7$-power velocity distribution for turbulent flow

because of the infinite value of the velocity gradient at the wall. As an explanation of this point it has been remarked[1] that in turbulent flow the velocities used in the above analysis are really mean values and that the actual flow is obtained by superimposing a fluctuating stream of relatively small amplitude upon them. In the immediate neighborhood of the bounding surface these fluctuations tend to disappear, leaving an extremely thin laminar sublayer underneath the turbulent one, with the result that the stress at the wall is not infinite but is accurately represented by Eq. (35). Experimental evidence supports this explanation (see footnote 2, page 319).

Problem 332. Determine the boundary-layer thickness at several points on a flat plate when placed in an airstream having a velocity of 25 ft. per sec. at a temperature of 20°C. The plate is 3 ft. long. Make the calculations for both laminar and turbulent flow at points 9 in. apart along the plate.

333. A flat plate 2.5 ft. long is immersed in a stream of water at 40°C. moving parallel to its surface with a velocity of 8 ft. per sec. Determine the shearing stress at the center and trailing edge of the plate and the drag coefficients for both laminar and turbulent flow.

334. Compute the drag coefficient for a flat plate at a Reynolds' number of 250,000 for both laminar and turbulent flow. In each case determine the effect of doubling (*a*) the length, (*b*) the velocity, (*c*) the kinematic viscosity.

168. The Transition from Laminar to Turbulent Flow.—In the previous articles the skin-friction drag and boundary-layer characteristics have been determined for a flat plate in the cases of completely laminar and completely turbulent flow. As already mentioned in Art. 160, there is a range between these two stages in which the forward portion of the boundary layer is laminar in character, while the remaining portion at the rear is turbulent. A solution of the resistance problem for this transition range has been given by Prandtl[2] by means of an ingenious combination of the results already obtained.

When the boundary layer on the flat plate becomes turbulent just at the trailing edge, the Reynolds' number corresponding to this condition and based on the length of the plate l has the critical value N_C. For further increases in the Reynolds' number

[1] PRANDTL and TIETJENS, *op. cit.*, pp. 78–80.

[2] PRANDTL, L., "Ergebnisse der Aerodynamischen Versuchsanstalt," Göttingen, vol. III, 1927, p. 1.

the transition point moves forward from the trailing edge and the boundary layer is partly laminar and partly turbulent as shown in Fig. 213. When the transition point is at a distance $x(0 \leqslant x \leqslant l)$ from the leading edge of the plate, the Reynolds' number based on the distance x must be equal to the critical value, that is,

$$\frac{\rho \bar{u} x}{\mu} = N_C$$

and for this part of the surface the drag coefficient is obtained by substituting N_C for N_R in Eq. (34*b*). The drag acting on it is

$$D_f \text{ (lam.)} = \frac{1.327 \rho \bar{u}^2 b x}{2\sqrt{N_C}} \tag{41}$$

The drag of the rear portion of the plate, over which the flow is turbulent, may be calculated by subtracting the drag of the

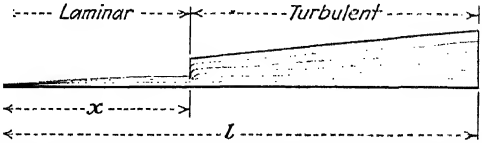


FIG. 213.—Boundary-layer contour in the transition range.

forward portion of the plate from that of the entire plate, both of these drags being computed on the basis of the turbulent-flow coefficient of Eq. (40*b*). The result of this calculation gives for the drag of the rear portion of the plate the value

$$D_f \text{ (turb.)} = \frac{0.074 \rho \bar{u}^2 b l}{2(N_R)^{1/5}} - \frac{0.074 \rho \bar{u}^2 b x}{2(N_C)^{1/5}} \tag{42}$$

and the sum of Eqs. (41) and (42) gives for the total drag

$$D_f = \frac{\rho \bar{u}^2 b l}{2} \left[\frac{1.327 \frac{x}{l}}{\sqrt{N_C}} + \frac{0.074}{(N_R)^{1/5}} - \frac{0.074 \frac{x}{l}}{(N_C)^{1/5}} \right] \tag{43}$$

The total drag coefficient is based on the area of one side and therefore has the value

$$C_f = \frac{0.074}{(N_R)^{1/5}} - \frac{0.074 \frac{x}{l}}{(N_C)^{1/5}} + \frac{1.327 \frac{x}{l}}{(N_C)^{1/2}}$$

Now the ratio $x/l = \rho\bar{u}x/\mu \div \rho\bar{u}l/\mu = N_C/N_R$, so that the above expression becomes

$$C_f = \frac{0.074}{(N_R)^{1/5}} - \frac{1}{N_R}\left| 0.074(N_C)^{4/5} - 1.327(N_C)^{1/2} \right| \tag{44}$$

The value of the critical Reynolds' number N_C depends on many factors such as the roughness of the surface of the plate, the shape of its leading edge and the initial turbulence of the stream of fluid before it reaches the plate. The most reliable experiments, which will be discussed in greater detail in the next article, indicate that for smooth plates this critical value may be in the neighborhood of 500,000. When this is introduced in Eq. (44), the result is approximately

$$C_f = \frac{0.074}{(N_R)^{1/5}} - \frac{1700}{N_R} \tag{45}$$

For other experimental conditions a different value of N_C may be obtained, but this affects only the coefficient of the second term of Eq. (45) and the necessary correction can easily be introduced by referring back to Eq. (44).

169. Experimental Data on Skin-friction Drag of Flat Plates.—Some knowledge of the validity of the boundary-layer theory and a justification for the various assumptions used may be obtained by comparing the theoretical results with values determined experimentally. The most complete data of this kind have been obtained by four German experimenters, Blasius, Wieselsberger, Gebers and Kempf, some being obtained in water and some in air. The results of their work, which cover the range of Reynolds' numbers from 10^5 to about 5×10^8, are shown graphically in Fig. 214. Logarithmic scales have been employed in plotting these data in order to facilitate comparison with the theoretical formulas. It will be recalled that the expressions for the drag coefficient in the cases of completely laminar and completely turbulent flow are of the form

so that

$$\log C_f = \log K - n \log N_R$$

The curve representing such an equation becomes a straight line of slope $-n$ when log C_f is plotted against log N_R. Three of the curves drawn in Fig. 214 represent the values of the drag coefficients as calculated from Eqs. (34*b*), (40*b*) and (45) for the laminar, turbulent and transition regimes, respectively. Except for a few scattered points, the agreement with the experimental results is excellent, although for Reynolds' numbers greater than about 3×10^7 certain deviations appear which indicate that the turbulent-flow formula gives values of C_f which are appreciably lower than the experimental ones.

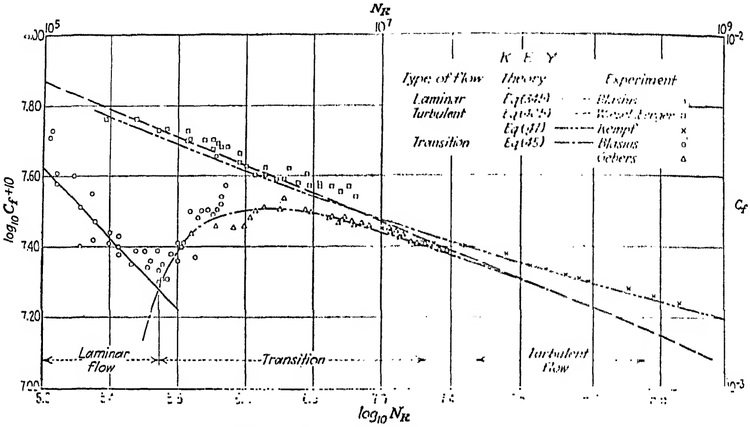


FIG. 214.—Variation of skin-friction drag coefficient with Reynolds' number for flat plates parallel to the flow.

Recent measurements of boundary-layer velocity-distribution profiles show that the $\frac{1}{7}$-power law which was used in the derivation of this formula is not entirely satisfactory for high Reynolds' numbers, but that more nearly correct results are obtained by using a velocity distribution of the form

$$= \bar{u}\left(\frac{y}{\delta}\right)^k$$

where the exponent k is itself a function of the Reynolds' number. The value $k = \frac{1}{7}$ is satisfactory up to the limit $N_R = 3 \times 10^7$, but beyond this point it gradually decreases, taking on the successive values of $\frac{1}{8}$, $\frac{1}{9}$, etc., as N_R increases. It is possible to take this change in k into account in determining the drag of the plate, but a more satisfactory method which does not

depend on an empirical determination of k as a function of N_R has been developed by von Kármán.[1] His treatment of the problem will not be given in detail here except to mention that a logarithmic expression for the velocity distribution is employed which according to Schoenherr[2] leads to the formula for the drag coefficient

$$\frac{0.242}{\sqrt{C_f}} = \log_{10}(N_R C_f) \tag{46}$$

This expression is rather clumsy to use in calculations but may be replaced by the simpler form suggested by Schlicting

$$C_f = \frac{0.455}{(\log_{10} N_R)^{2.58}} \tag{47}$$

which agrees closely with the values computed from Eq. (46) in the range $10^6 \leqq N_R \leqq 10^9$.

In connection with this discussion of theoretical and experimental values for flat-plate resistance coefficients, it might be well to mention the effect of surface conditions. In general it may be stated that in the case of laminar flow roughness has little effect on the resistance as long as the dimensions of the hills and valleys, measured normal to the surface, are small enough so that the projections on the surface do not penetrate the boundary layer. Roughness on the forward portion of the plate will cause the transition to turbulent flow to begin at a lower critical Reynolds' number than for a smooth plate. When the roughness is so coarse that the projections on the surface extend outside the boundary layer, the body ceases to be a flat plate in the sense considered here and there is a considerable increase in resistance due to separation and wake formation.

The shape of the nose of the plate also has some effect on the critical Reynolds' number in that, when the nose is sharpened, the transition to turbulent flow will not set in so quickly as with a rounded nose. Wieselsberger's results, shown in Fig. 214, were obtained with a relatively blunt round-nosed plate. This explains why his points for the lower Reynolds' numbers remain on the turbulent-flow curve.

[1] See footnote 2, p. 319.

[2] SCHOENHERR, K. E., Resistance of Flat Surfaces Moving through a Fluid, *Trans. Soc. Naval Arch. and Marine Eng.*, vol. 40, p. 279, 1932.

Example.—A flat plate 1 ft. long is placed in a stream of water at 40°C. with the flow parallel to its surface and in the direction of the length. The velocity is 11.73 ft. per sec. Find the location of the point where the boundary-layer flow changes from laminar to turbulent and determine the thickness of the layer at several points on the plate.

Solution.—For water at 40°C. the absolute viscosity as given by Table V, page 177, is

$$\mu = 0.653 \times 10^{-2} \text{ poises} = 1.368 \times 10^{-5} \text{ slugs/ft. sec.}$$

The density is $\rho = w/g = 62.4/32.2 = 1.94$ slugs/cu. ft. Then the Reynolds' number is

$$N_R = \frac{\rho \bar{u} l}{\mu} = \frac{1.94 \times 11.73 \times 1}{1.368 \times 10^{-5}} = 16.63 \times 10^5 = 1,663,000$$

For this value of N_R the boundary-layer flow is in the transition range. Assuming that $N_C = 500,000$ the distance from the leading edge of the plate to the transition point is given by the relation

$$\frac{x}{l} = \frac{N_C}{N_R} \qquad \text{o} \qquad x = l\frac{N_C}{N_R} = \frac{1 \times 500,000}{1,663,000} = 0.30 \text{ ft.}$$

From $x = 0$ to 0.3 ft. the boundary layer is laminar and its thickness is determined by Eq. (32) with N_R put equal to N_C. Then

$$\delta_l = 5.48\sqrt{\frac{xl}{N_C}} = 5.48\sqrt{\frac{0.3}{500,000}}\sqrt{x} = 0.00424\sqrt{x}$$

For the turbulent portion of the boundary layer the thickness is the same as if the flow were turbulent over the entire surface. Then Eq. (37) applies and

$$\delta_t = 0.377\left(\frac{\mu}{\rho \bar{u} x}\right)^{1/5} x = 0.377\left(\frac{\mu}{\rho \bar{u} l}\right)^{1/5}\left(\frac{l}{x}\right)^{1/5} x$$

$$= \frac{0.377 l^{1/5}}{N_R^{1/5}} x^{4/5} = \frac{0.377 \times 1}{(16.63 \times 10^5)^{1/5}} x^{4/5} = 0.0215 x^{4/5}$$

Values of δ are given in the following table:

x, ft.	δ-Laminar		x, ft.	δ-Turbulent	
	Ft.	In.		Ft.	In.
0	0	0	0.3	0.0082	0.0984
0.1	0.00134	0.0161	0.5	0.0124	0.149
0.2	0.00190	0.0228	0.75	0.0171	0.210
0.3	0.00232	0.0278	1.00	0.0215	0.258

Problem 335. The conditions of flow past a flat plate correspond to a Reynolds' number of 750,000. Compute the drag coefficient, assuming (*a*) that the boundary layer is completely turbulent, (*b*) that the flow is in the transition range with $N_C = 500,000$.

336. (*a*) An increase in the surface roughness of a flat plate causes the critical Reynolds' number to drop to 250,000. What is the equation for C_f in the transition range? (*b*) Compute the drag coefficient for $N_R = 10^6$, when $N_C = 500{,}000$ and when the plate has the roughness indicated in (*a*).

337. Determine the boundary layer thickness at quarter-points on a smooth plate 10 ft. long for which $N_C = 500{,}000$ when (*a*) $N_R = 100{,}000$, (*b*) $N_R = 1{,}000{,}000$, (*c*) $N_R = 10{,}000{,}000$ and the flow is assumed to be completely turbulent.

338. The boundary layer on a flat plate is completely turbulent. Determine the values of C_f by means of the $\frac{1}{7}$-power-law formula and by Schlieting's formula for $N_R = 10^7$ and 10^9. Assuming Schlieting's formula to be correct, what are the errors in C_f, using the $\frac{1}{7}$-power-law formula?

170. Eddy-making Resistance. The mechanism of wake formation and the production of resistance as a result of the

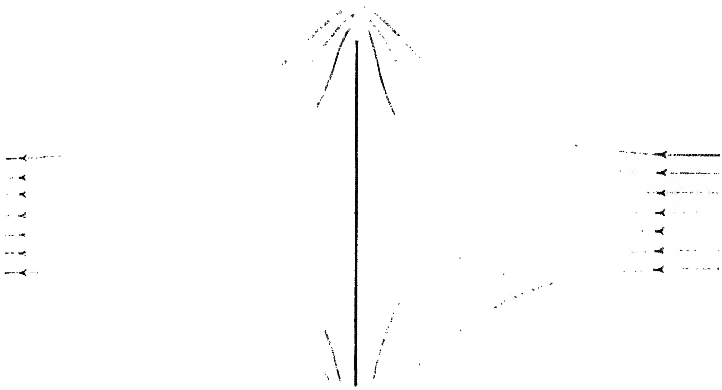


FIG. 215.—Theoretical two-dimensional flow past a normal flat plate without wake formation.

formation of eddies have been explained in detail, but theoretical methods for the calculation of the magnitude of this drag are far from being satisfactory, and for most bodies it is necessary to determine this resistance by experimental methods. The theory of perfect fluids, when separation is not taken into account, is of little value. For example, the two-dimensional flow past an infinitely long plate normal to the direction of the stream is, according to this theory, of the character illustrated in Fig. 215. Because of the symmetry of the flow with respect to the line of the plate, there can be no resistance opposing its motion. This flow is an example of the paradox of D'Alembert.

The type of flow illustrated in Fig. 215 is not even approximated by the flows of real fluids except possibly in the case of an extremely low Reynolds' number corresponding to the creeping type of motion. At the more usual values of N_R separation takes place at the edges of the plate and a strong eddying wake is

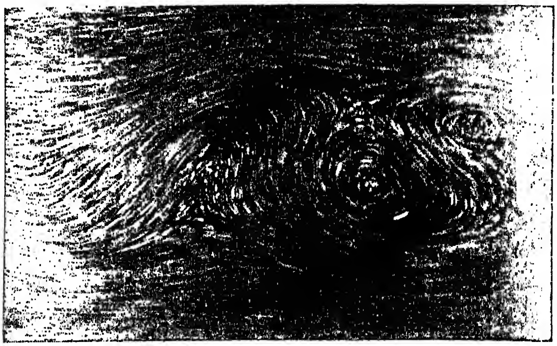


FIG. 216.—Flow normal to a flat plate. (*Prandtl and Tietjens, "Applied Hydro- and Aeromechanics," McGraw-Hill Book Company, Inc., New York*, 1934.)

formed as shown in Fig. 216. A theoretical method based on the perfect fluid but taking into account this wake formation was developed by Helmholtz and Kirchhoff and has been applied both to the normal plate and to plates inclined at fairly large angles to the direction of the flow. In the former case the flow in the wake is idealized to give the streamlines shown in Fig. 217. Surfaces of discontinuity are assumed to start at the sharp edges of the plate and to extend indefinitely downstream behind it. These surfaces are stable in form and enclose a mass of fluid which is at rest and at the same pressure throughout as the stream far ahead of the plate. This portion of the fluid is sometimes called the "dead-water" region. The eddies formed in the actual wake, shown in Fig. 216, are not taken into account in this theory, nor is any consideration given to the fact that there

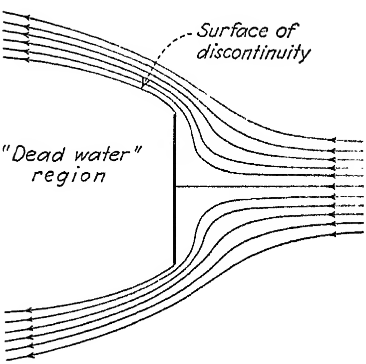


FIG. 217.—Theoretical flow past a normal flat plate with "dead-water" wake.

is an appreciable reduction in pressure behind the plate which increases the drag. The theoretical drag coefficient, which has the value

$$C_D = \frac{2\pi}{4 + \pi} = 0.88 \tag{48}$$

is thus appreciably lower than experimental values. For a plate of infinite length with two-dimensional flow past any cross section, the drag coefficient is approximately equal to 2.0. In the case of rectangular plates of finite length the drag coefficient is a function of the ratio of the length and breadth as shown by the results plotted graphically in Fig. 218. The reduction in

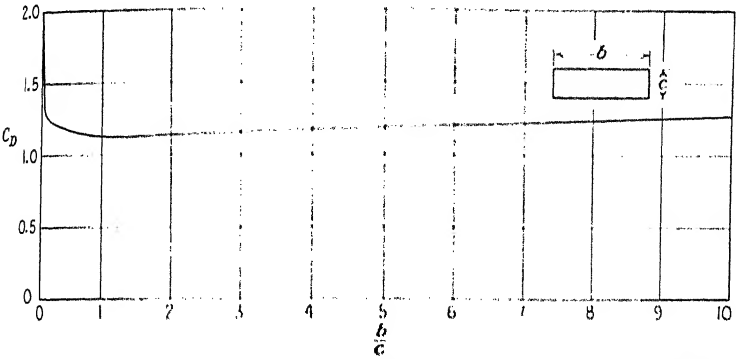


FIG. 218.—Variation of drag coefficients for normal plates with ratio of length to breadth.

drag coefficient for a rectangular plate as compared with one infinitely long is due to the fact that, when the length is finite, fluid flows around the narrow ends into the wake and increases the pressure in that region. It is obvious that the drag coefficients for plates whose length-breadth ratios are reciprocals are identical in value. For circular disks the drag coefficient is about 1.10 at high Reynolds' numbers.

The effect of variations in Reynolds' number on the resistance of normal plates is of considerable interest. Although most work of this kind has been concerned with circular disks, the results are very similar to those obtained with plates of other forms. Values of the drag coefficient for the circular disk at Reynolds' numbers based on the diameter and covering the range from $N_R = 0.4$ to 10^6 are shown in Fig. 219. For extremely

low values of N_R the drag coefficient is large but decreases rapidly as the Reynolds' number is increased. Beyond a value of N_R equal to about 10^3, the drag coefficient is practically constant and equal to 1.10. The results shown by this curve are in accordance with the statements given in Art. 158. When C_D is plotted as a function of N_R on logarithmic scales as in Fig. 219, the curve for very low Reynolds' numbers, represented by the equation $C_D = K/N_R$, is a straight line having a slope equal to -1. In the case of the disk this relation holds only for Reynolds' numbers up to about 2. It is interesting to note that in this range the

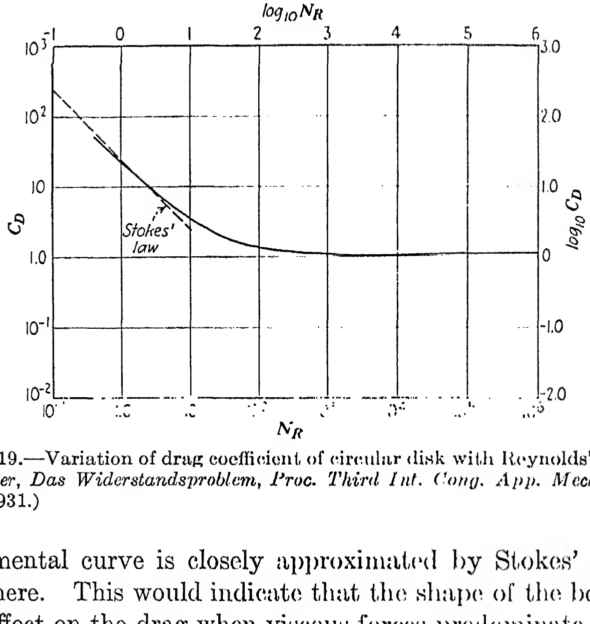


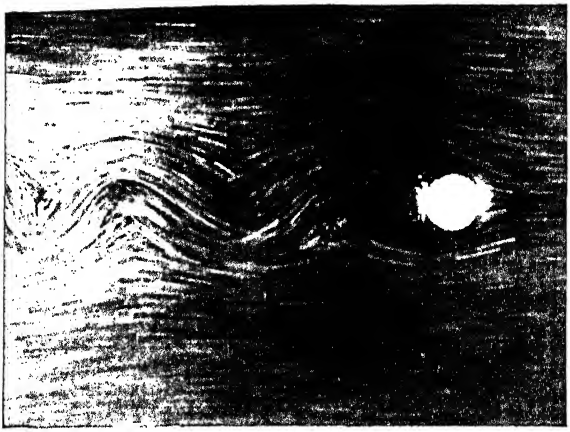
FIG. 219.—Variation of drag coefficient of circular disk with Reynolds' number. (*F. Eisner, Das Widerstandsproblem, Proc. Third Int. Cong. App. Mech.* (*Stockholm*), 1931.)

experimental curve is closely approximated by Stokes' law for the sphere. This would indicate that the shape of the body has little effect on the drag when viscous forces predominate.

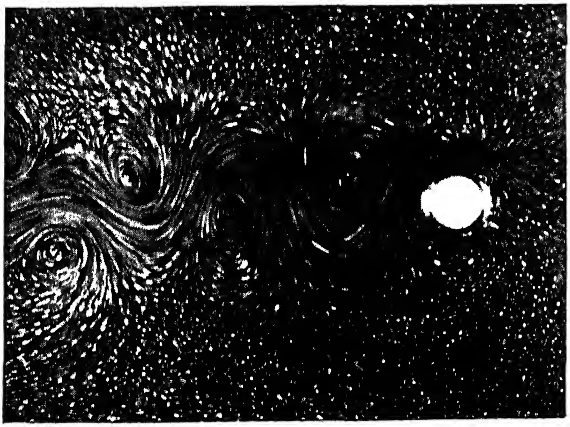
For values of N_R above 10^3 the drag coefficient is independent of N_R and the drag is proportional to the square of the velocity. In this range the inertia forces predominate in determining the character of the flow. Separation, once begun, always takes place at the sharp edge of the disk so that there is little change in the wake with further increases in Reynolds' number.

The detailed nature of the wake behind such bodies as normal plates and circular cylinders was first investigated experimentally by Bénard, who demonstrated the existence of a definite arrange-

ment of eddies or vortices in the region behind the body. A theoretical investigation by von Kármán showed that, when the vortices are arranged periodically in two rows parallel to the



(a)



(b)

FIG. 220.—Vortex trail behind a circular cylinder. (a) Camera at rest with respect to cylinder. (b) Camera at rest with respect to undisturbed fluid. (*Prandtl and Tietjens, "Applied Hydro- and Aeromechanics," McGraw-Hill Book Company, Inc., New York*, 1934.)

direction of motion as shown for the circular cylinder in Fig. 220, the system is stable provided the centers of the vortices are staggered and the ratio between the spacing of the rows and

the longitudinal spacing of the vortices is equal to 0.281. An application of the momentum theorem gives a value for the resistance, due to the formation of this vortex trail, which depends on the ratio of the velocities of translation of the body and the vortices and also on the ratio of the longitudinal spacing of the eddies to some characteristic length of the body. The complete solution is obtained by an experimental determination of these ratios and appears to give results which agree reasonably well with test data.[1]

The formation of a vortex trail behind a circular cylinder is sometimes found to be the cause of undesirable vibrations of structures having such a shape. The eddies in the wake separate from the rear surface of the cylinder at a definite frequency and, if this value is in the neighborhood of the natural frequency of the structure, a condition of resonance may result in which the vibration of the structure may actually become dangerous. This periodic vortex formation has been found to be responsible for the vibration of electric transmission lines and gas stacks which are exposed to the action of natural wind currents. In some cases the vibration may occur at very low speeds for which the average wind force on the structure is practically insignificant.

Eddy formation of this type also occurs behind lifting vanes at high angles of attack and is often an important factor in connection with the vibration of turbine and propeller blades and the flutter of airplane wings.

Problem 339. Determine the resistance of a signboard 8 by 20 ft. in a wind current of 35 m.p.h., standard density, and in a direction normal to the surface of the sign (*a*) when mounted far off the ground, (*b*) when the 20-ft. edge rests on the ground. Neglect boundary layer at ground.

340. The drag coefficient of a parachute is approximately the same as that of a circular disk of the same diameter. What is the rate of descent of a man with a 16-ft.-diameter parachute if the total weight is 200 lb.? Assume drag equal to weight and air of standard density.

341. Certain specifications for parachutes require that the velocity of descent shall be the same as that attained in jumping from a height of 6 ft. What diameter parachute would be required for a total load of 200 lb.? Assume drag coefficient equal to that of circular disk of same diameter and air of standard density.

[1] For a detailed discussion of this subject, see Prandtl and Tietjens, *op. cit.*, pp. 130–136; H. Lamb, "Hydrodynamics," 5th ed., pp. 208, 212, Cambridge University Press. 1926.

171. Resistance of Bodies of Revolution. The modern engineer is frequently called upon to determine the resistance of such shapes as the body of an airship, airplane, submarine or automobile. Efficient operation of such vehicles requires that the resistance be made as low as possible. The determination of the resistance of such forms is usually a rather complicated problem but light may be shed on it by a consideration of the resistance of bodies of revolution. Such bodies are produced by revolving a symmetrical contour around its axis.

The problem of finding bodies of revolution of minimum resistance is an exceedingly complicated one and must be treated experimentally in most cases in order to obtain quantitative

Usually the problem is to find a shape having a certain

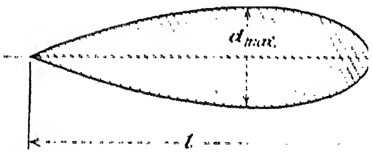


FIG. 221. Longitudinal section of a streamlined body of revolution.

specified length, projected area or volume and the lowest possible drag. In general it may be said that such a shape should have a well-rounded nose rather than a sharp one since the motion of a body through a fluid is not a cutting action. The point of maximum diameter should be back from the nose a distance of three to five tenths of the length of the body, and the ratio of the length to the maximum diameter should, if possible, be between 2 and 3. The section should have a gradually tapering tail so as to avoid unduly high pressure gradients which tend to produce separation. For the same reason sharp corners should be avoided all along the section except the rear, which, if practical, should be brought to a sharp point. These statements indicate that an ideal streamlined body should have a longitudinal section of the type illustrated in Fig. 221.

Direct comparisons of the drag coefficients of bodies of revolution of different sections are difficult to make because of the variation of these coefficients with Reynolds' number. The most exhaustive studies in this connection have been made on spheres and the results of a large number of such investigations are shown in Fig. 222. The extremely low values of N_R cor-

respond to the viscous type of flow in which Stokes' law is valid. This solution as given by Eq. (11), $C_D = 24/N_R$, is also included in Fig. 222, and it appears that this equation agrees with the experimental data only for values of N_R up to about 0.4. For higher values of N_R the inertia forces become more important and the drag coefficient decreases less rapidly with the Reynolds' number, approaching a practically constant value in the range from 10^3 to 10^5.

Beyond $N_R = 10^3$ the flow is characterized by the existence of a boundary layer, and the sudden drop in C_D between the values of N_R of 10^5 and 10^6 is due to a change in the boundary-

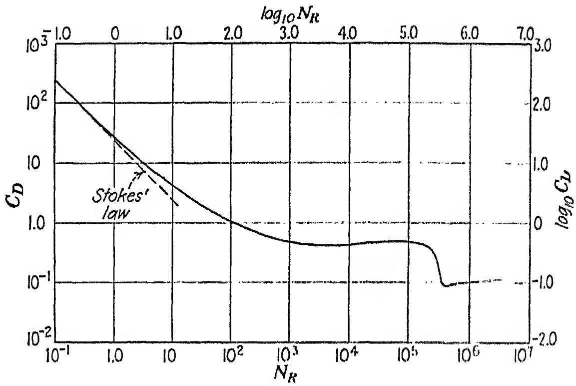
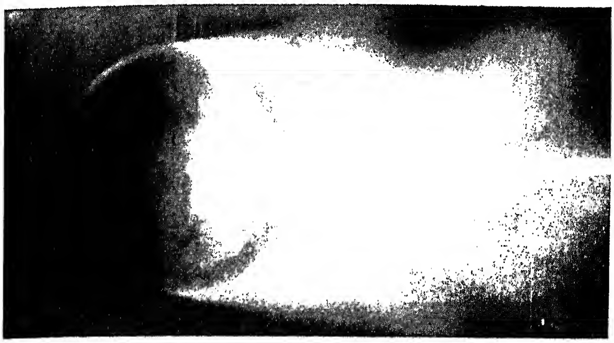


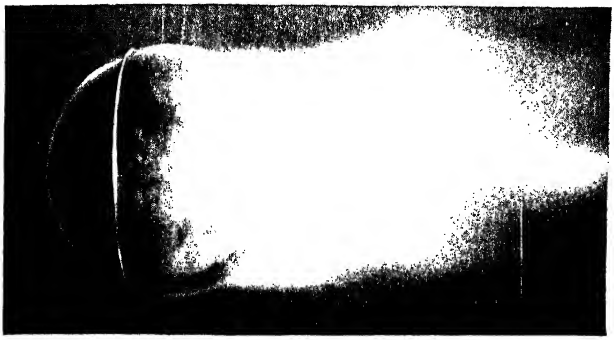
FIG. 222.—Variation of drag coefficient of a sphere with Reynolds' number. (*F. Eisner, Das Widerstandsproblem, Proc. Third Int. Cong. App. Mech. (Stockholm)*, 1931.)

layer flow from a laminar to a turbulent character. This transition is somewhat analogous to the change in the drag of a parallel flat plate under similar circumstances but in that case the transition resulted in an increase in the drag coefficient. The drag of the sphere in this range of Reynolds' numbers consists of skin-friction and eddy-making drags, the latter being the predominant factor. When the boundary-layer flow changes from laminar to turbulent, the separation point at the rear of the sphere is moved backward as explained in Art. 164. The wake then becomes narrower and the eddy-making drag is decreased. This transition is also accompanied by an increase in skin friction but the combined effect is a marked reduction in the total drag coefficient as shown in Fig. 222.

As in the case of the parallel flat plate, an increase in surface roughness of the sphere or of the initial turbulence of the stream causes the transition to take place at a lower value of N_R. Smoke pictures made by Wieselsberger of the flow of air past a sphere, illustrating the effect of increased roughness, are shown in Fig. 223. In Fig. 223*a* is shown the flow past a smooth sphere at a



(*a*)



(*b*)

FIG. 223.—Effect of surface roughness on flow past a sphere. (*a*) Smooth sphere. (*b*) Smooth sphere with wire. (*Prandtl and Tietjens, "Applied Hydro- and Aeromechanics," McGraw-Hill Book Company, Inc., New York,* 1934.)

Reynolds' number below the critical. Here the boundary layer is laminar, separation takes place at the ends of the vertical diameter and the wake is wide. If a very fine wire is placed around the sphere in a plane parallel to and slightly forward of the plane of the vertical great circle, the boundary layer is artificially changed to a turbulent flow and the picture shown in Fig. 223*b* is obtained. A similar picture could be obtained

by increasing the Reynolds' number to a value higher than the critical.

The curve of Fig. 222 does not extend far beyond the transition point but the indications are that, once the boundary layer has become completely turbulent, the form of the wake does not change appreciably and the drag coefficient is again practically constant. The critical Reynolds' number for a sphere is usually defined[1] as that value of N_R for which the drag coefficient is equal to 0.3.

The sphere offers an interesting example for studying the nature of the flow around bodies of revolution but it is not a satisfactory shape where low resistance is desired. For such cases a more elongated form is preferred. By extending the rear portion of the sphere so as to produce a body of revolution with a gradually tapering tail, the resistance may be materially decreased. When the body is properly "streamlined" the wake behind it is almost completely eliminated and the drag is practically all due to skin friction. It has been proved experimentally that for low drag forms, such as airship hulls, there is no separation and consequently no eddy-making resistance so long as the axis of revolution is parallel to the direction of the motion of the hull.

The effect of modifying the spherical shape may be studied by considering the drag of ellipsoids of revolution of different proportions. In this connection the ratio of the major and minor axes is important. For a body of revolution of any shape having a longitudinal section such as that shown in Fig. 221, this quantity is known as the fineness ratio and is equal to the expression

$$f = \frac{\text{length}}{\text{maximum diameter}} = \frac{l}{d_{\text{max.}}}.$$

Experimental values of the drag coefficients for a series of ellipsoids of different fineness ratios plotted against Reynolds' number are shown in Fig. 224. The values of C_D are based on the area of the largest cross section and the length used in determining N_R is the diameter of this section. The curves for the sphere, circular disk and a typical airship hull are also included in Fig. 224.

[1] DRYDEN, H. L., and A. M. KUETHE, The Effects of Turbulence on Wind Tunnel Measurements, *NACA Tech. Rept.* 342.

In general the curves for the ellipsoids are similar in shape to the sphere drag curve. There is a transition from a laminar to a turbulent boundary layer which causes a reduction in the value of C_D but, as the fineness ratio increases, the change in C_D becomes less marked and the critical Reynolds' number at which it occurs decreases. The normal circular disk has a fineness

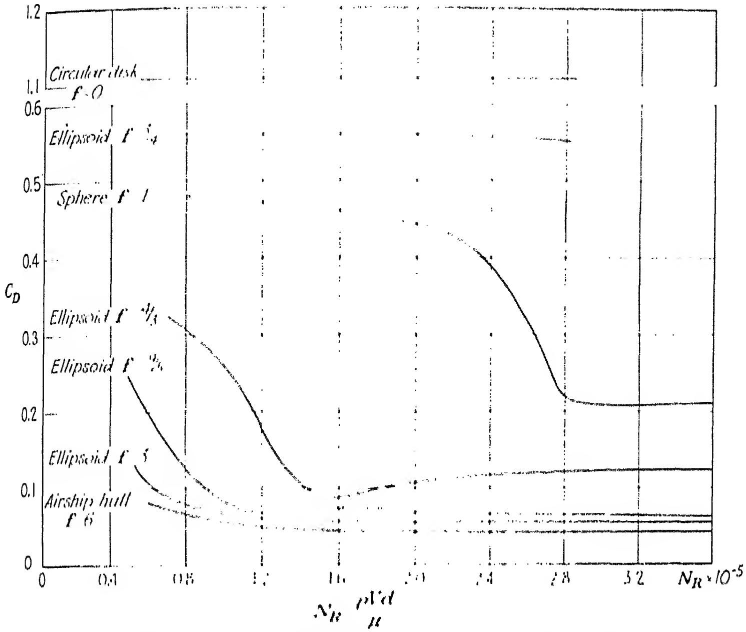


FIG. 224.—Variation of drag coefficients for bodies of revolution with Reynolds' number. (*W. Müller, "Mathematische Strömungslehre," Julius Springer, Berlin,* 1928.)

ratio of zero and the fact that its drag coefficient is practically constant in the range of Reynolds' numbers considered might be taken as evidence that its critical Reynolds' number is infinitely large. A comparison of the drag coefficients for the sphere with those of Fig. 222 shows a large discrepancy at Reynolds' numbers above the critical. This may be due to differences in experimental technique, surface roughness and initial turbulence of the streams. Although the curve of Fig. 222 represents the usually accepted values of C_D for the sphere, the data given in Fig. 224 are nevertheless of value in presenting a

picture of the relative resistance of bodies of different fineness ratios.

Example.—Determine the drag force acting on the following bodies of revolution when placed in an airstream having a velocity of 100 ft. per sec.:

Circular disk.

Sphere.

Ellipsoids $- f = \frac{3}{4}, \frac{4}{3}, \frac{9}{5}$ and 3.

Airship hull $- f = 6$.

In each case the maximum diameter is 6.02 in. and the axis of symmetry is parallel to the direction of the stream.

Using the circular disk as a standard, determine the percentage reduction in drag due to streamlining for each body.

Solution.—Assuming standard air, the density is $\rho = 0.002378$ slugs per cu. ft. and the kinematic viscosity from Table IV, page 176, is

$$\nu = 1.566 \times 10^{-4} \text{ ft.}^2/\text{sec.}$$

Then the Reynolds' number based on the maximum diameter is

$$N_R = \frac{\rho V d}{\mu} = \frac{Vd}{\nu} = \frac{100 \times \frac{6.02}{12}}{1.566 \times 10^{-4}} = 3.2 \times 10^5$$

The drag force, using coefficients based on the area of the maximum cross section, is

$$D = C_D \frac{\rho V^2}{2} A \qquad \frac{C_D \times 0.002378 \times (100)^2 \times \frac{\pi}{4}\left(\frac{6.02}{12}\right)^2}{2} = 2.35 C_D$$

The values of C_D are taken from Fig. 224. The remaining computations are conveniently put in tabular form as follows:

Body	C_D	D, lb.	$D_{\text{Disk}} - D$	$\frac{(D_{\text{Disk}} - D)}{D_{\text{Disk}}}$
Disk........	1.108[1]	2.60	0	0
Sphere......	0.209	0.49	2.11	81.2
Airship hull.	0.044	0.10	2.50	96.2
Ellipsoids:				
$f = \frac{3}{4}$	0.541[1]	1.27	1.33	51.2
$f = \frac{4}{3}$	0.123	0.29	2.31	88.9
$f = \frac{9}{5}$	0.064	0.15	2.45	94.3
$f = 3$	0.056	0.13	2.47	95.0

[1] Extrapolated.

172. Resistance of Cylinders.—In studying the resistance of cylinders of different cross sections, it is convenient to consider

first only those shapes which have a section with at least one axis of symmetry. The problem is further simplified by placing this axis parallel to the direction of flow and having the cylinder generators perpendicular to that direction.

Most of the experimental data available are concerned with circular cylinders, and the variation of C_D with Reynolds' number is shown in Fig. 225. This curve is similar in character to that for the sphere shown in Fig. 222. At low values of N_R the drag coefficient varies inversely with N_R while at higher

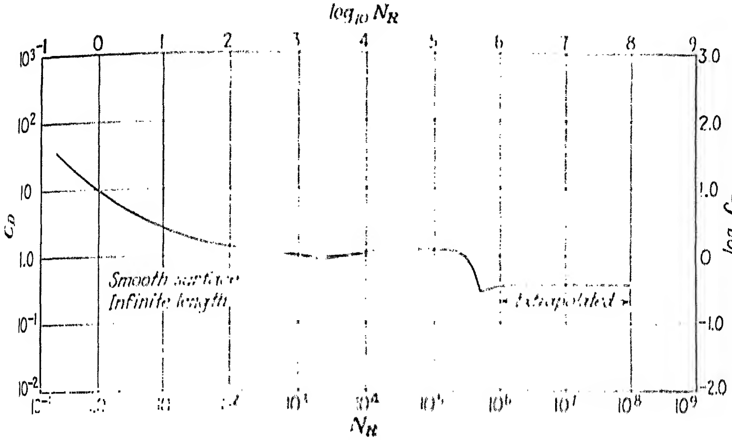


FIG. 225. Variation of drag coefficient with Reynolds' number for a circular cylinder. (*F. Eisner, Das Widerstandsproblem, Proc. Third Int. Cong. App. Mech. (Stockholm)*, 1931.)

values C_D is practically constant up to N_R equal to about 10^5. Between Reynolds' numbers of 10^5 and 10^6 the drag coefficient shows the characteristic drop due to the transition from laminar to turbulent boundary-layer flow.

When it is desired to obtain a cylinder of low resistance, the cross section should be similar in shape to the contour shown in Fig. 221. As in the case of bodies of revolution the drag is closely related to the fineness ratio. According to Diehl,[1] the ratio of the drag coefficient to its minimum value varies with fineness ratio as shown in Fig. 226. More recent experimental data show a considerable disagreement with Diehl's results.

[1] DIEHL, W. S., "Engineering Aerodynamics," p. 74, Fig. 46, Ronald Press Company, New York, 1928.

Tests[1] made by the National Advisory Committee for Aeronautics give the results indicated by the curve marked "NACA" in Fig. 226. The discrepancies may be due to differences in experimental technique. Similar data for airship hulls are also included in Fig. 226.

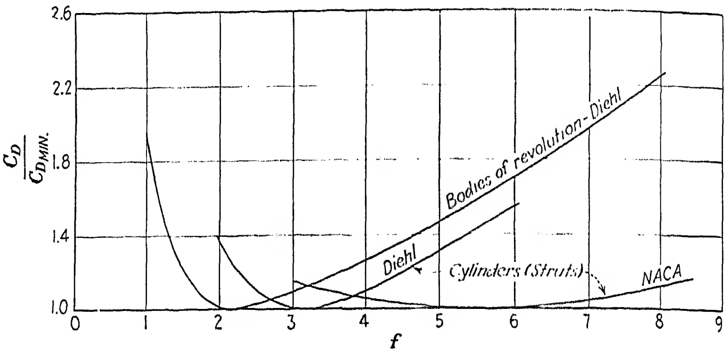


FIG. 226.—Variation of $C_D/C_{D_{\min}}$ with fineness ratio for cylinders and bodies of revolution.

An empirical formula[2] for cylinders having sections of the type shown in Fig. 221, which is satisfactory for fineness ratios from infinity to about 3, is

$$C_D = 0.0065 + 0.125\left(\frac{1}{f^2}\right) \tag{49}$$

where f is the fineness ratio. This expression is based on experimental data obtained from tests on symmetrical airfoil sections at Reynolds' numbers of about 3×10^6. Because such sections closely resemble the parallel flat plate, the drag coefficient in Eq. (49) is based on the area projected on a plane parallel to the direction of flow rather than normal to it and the Reynolds' number is computed by using the length of this projection in the direction of motion.

Cylindrical bodies of rectangular cross section are shapes of high resistance, being comparable to the normal flat plate. While such shapes should be carefully avoided in machines

[1] BIERMANN, D., and W. J. HERRNSTEIN, JR., The Interference between Struts in Various Combinations, *NACA Tech. Rept.* 468.

[2] UPSON, R. H., and M. J. THOMPSON, The Drag of Tapered Cantilever Airfoils, *J. Aero. Sci.*, vol. 1, October, 1934.

designed for high-speed transportation, they are unavoidable in buildings where the question of wind resistance is an important one, particularly in extremely high structures. Obviously the use of a streamlined section for a skyscraper would be impracticable even if the structural difficulties were surmounted because of the continually changing direction of the natural wind currents. Some knowledge of the resistance and pressure distribution on cylinders with square and rectangular cross sections is therefore of value.

Because of the presence of sharp corners, separation always occurs at the edges of the upstream face of the prism, and the variation of drag with Reynolds' number is practically negligible. In most cases the coefficients for normal flat plates are employed, although there is evidence that an increase of the length of the cross section in the direction of motion tends to decrease the drag coefficient somewhat. Eiffel[1] found experimentally that, for a right circular cylinder with its base normal to the direction of flow, the drag varies with length-diameter ratio as shown by the following data. The actual drag coefficient is not given here but only its ratio to that of the normal circular disk. These results

l/d	0	0.5	1.0	1.5	2.0	3.0	4.0	
$\dfrac{C_D}{C_D\text{ (disk)}}$	1.0	1.0	0.83	0.80	0.77	0.77	0.78	0.89

show that an increase in length produces an appreciable reduction in drag for length-diameter ratios up to about 3. This is undoubtedly due to a reduction in eddy-making drag. When the length-diameter ratio is greater than 3, the surface area of the cylinder is so large that any reduction in eddy-making drag is more than balanced by an increase in skin friction. The total drag then shows an increase.

Example.—Compute the drag force per foot of length for a round strut 1 ft. in diameter and for a streamlined strut of symmetrical cross section having a maximum thickness of 1 ft. and a fineness ratio of 3. Both struts are placed in an airstream of 200 ft. per sec. velocity, the axis of symmetry of the cross section of the streamlined strut being parallel to the direction of the stream.

[1] EIFFEL, G., "The Resistance of the Air and Aviation," p. 66, Archibald Constable & Company, Ltd., London, 1913.

Solution.—For standard air $\rho = 0.002378$ slugs per cu. ft. and

$$\nu = 1.566 \times 10^{-4} \text{ ft.}^2 \text{ per sec.}$$

(Table IV, page 176). Then for the round strut

$$N_R = \frac{Vd}{\nu} = \frac{200 \times 1}{1.566 \times 10^{-4}} = 1.277 \times 10^6$$

From Fig. 225 $C_D = 0.365$ so that

$$D = C_D \frac{\rho V^2}{2} A = \frac{0.365 \times 0.002378 \times (200)^2 \times 1}{2} = 17.35 \text{ lb./ft.}$$

For the streamlined strut $t_{\max.} = 1$ ft. and, since $f = 3$, the chord is $l = 3$ ft. The Reynolds' number based on the chord is therefore

$$N_R = \frac{Vl}{\nu} = \frac{200 \times 3}{1.566 \times 10^{-4}} = 3.83 \times 10^6$$

Assuming that Eq. (49) holds for this value of N_R, the drag coefficient is

$$C_D = 0.0065 + \frac{0.125}{f^2} = 0.0065 + \frac{0.125}{9} = 0.0204$$

Since this coefficient is based on the area projected in the plane of the chord, the drag is

$$D = C_D \frac{\rho V^2}{2} A = \frac{0.0204 \times 0.002378 \times (200)^2 \times 3}{2} = 2.91 \text{ lb./ft.}$$

Problem 342. A long circular cylinder 6 in. in diameter moves through water at 20°C. in a direction perpendicular to its axis. Determine the drag force per foot of length at velocities between 2 ft. per sec. and 25 ft. per sec., taking enough values so as to determine definitely the drag in the critical range. Plot a curve of D versus V.

343. Determine the drag of a strut 12 ft. long and of a 10-in. chord placed in a stream of standard air. The fineness ratio is 4 and the Reynolds' number is 3×10^6. What is the velocity?

344. The diameter of a sphere and the maximum diameter of an ellipsoid of revolution are 5 in. The fineness ratio of the ellipsoid is 3. Determine the drags of these bodies in standard air at a velocity of 75 m.p.h.

345. What is the wind force on the face of a building 350 ft. high and 50 ft. wide if its drag coefficient is the same as for a rectangular plate? The air is standard and the velocity is 60 m.p.h.

346. What is the limiting velocity of the flow of an oil ($\mu = 0.0025$ slugs per ft. sec., specific gravity $= 0.79$) past a sphere of 0.001 in. diameter for which Stokes' law is valid? What are the drag coefficient and the drag force under these conditions?

347. An airship hull having a maximum cross section 85 ft. in diameter and a fineness ratio of 6 travels through standard air at 90 m.p.h. What are the drag force and power required to maintain this speed?

173. Resistance of Lifting Surfaces.—The lifting vane or airfoil was discussed in Chap. VII and the nature of the lift force was studied in detail. The drag of such bodies, which are nonsymmetrical cylinders, behaves in much the same way as that for the symmetrical shapes discussed in Art. 172 of the present chapter, but the problem is complicated by the variation of

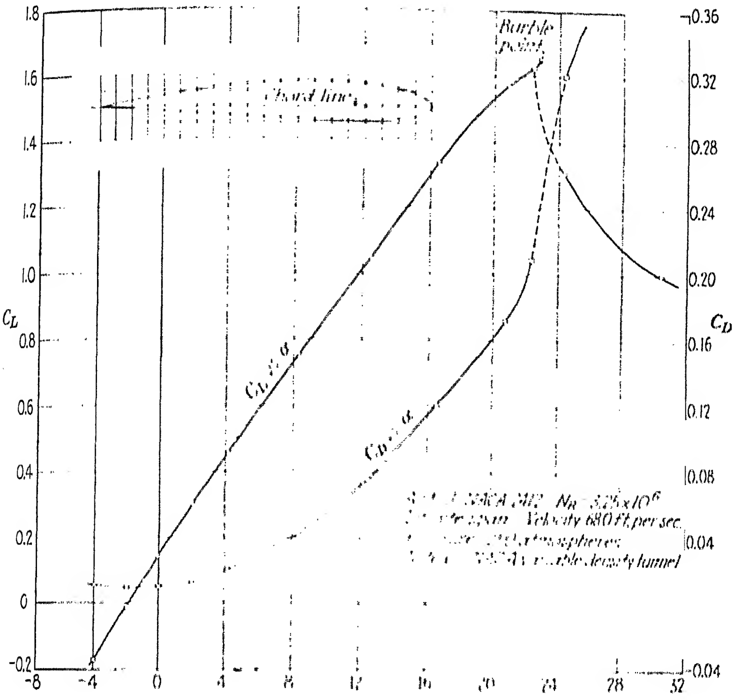


FIG. 227. Variation of lift and drag coefficients of the NACA 2412 airfoil with angle of attack. (*E. N. Jacobs, K. E. Ward and R. M. Pinkerton, The Characteristics of 78 Related Airfoil Sections from Tests in the Variable-density Wind Tunnel, NACA Tech. Rept.* 460.)

drag with inclination of the surface. When the vane is long so that end effects are minimized, there is an angle of minimum drag which is usually close to the position of zero lift. This minimum drag is increased by increases in either the maximum thickness or the curvature of the section but for most well-designed sections it consists primarily of skin-friction drag. As the inclination of the section is increased, the drag increases approximately with

the square of the angle, this change being similar in nature to the effect of an increase in fineness ratio for a symmetrical cylinder. Finally, when the inclination is sufficiently high, separation occurs on the upper surface at a point forward from the trailing edge and on the lower surface at the trailing edge. This separation is known as the burbling of flow and the position at which it occurs is known as the burble point. This separation is responsible for the fact that the lift of the airfoil reaches a maximum in the neighborhood of this position, while at the same time the strong eddying wake which is formed causes a sharp increase in the drag. The values of the drag coefficient for a typical section based on its plane area are shown in Fig. 227 for angles varying from zero lift to beyond the maximum lift. For convenience the lift curve has also been plotted in this diagram.

174. Induced Drag of Lifting Vanes. For the vane of infinite span the drag is made up of skin-friction and eddy-making

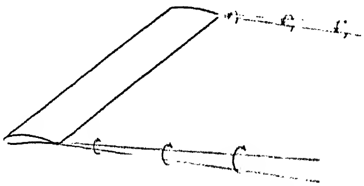


FIG. 228.—Tip vortices on an airfoil of finite span.

components but when the span is finite there is an additional drag produced by the action of the lift. For any cross section of a vane producing an upward lift, it is known that there is a decrease in pressure over the upper surface and an increase on the lower. In the case of a finite span these two pressures tend to equalize each other in the neighborhood of the wing tip and fluid flows around the tip from the bottom of the airfoil to the top. This flow produces a vortex which is left behind the wing as it moves forward through the fluid, the direction of rotation of the vortex being as shown in Fig. 228. The effects of the vortices, one being formed at each tip, are to induce a downward velocity at the wing which has the principal result of decreasing the effective angle of attack. If the lift is to be maintained constant, then the apparent angle of attack must be increased by an amount

$$\epsilon = \frac{w}{V} \tag{50}$$

where w is the average induced velocity and V the forward velocity and their ratio is assumed to be small. These conditions are shown in detail in Fig. 229. For the wing of finite span the true lift is measured perpendicular to the true velocity vector so that its vertical component is

$$L = L_\infty \cos \epsilon \approx L_\infty \tag{51}$$

There is also a horizontal component of the lift, L_∞, which is known as the induced drag and has the value

$$D_i = L_\infty \sin \epsilon \approx L_\infty \frac{w}{V} \tag{52}$$

The drag of the vane when the span is infinite is known as the profile drag because it depends only on the profile or cross

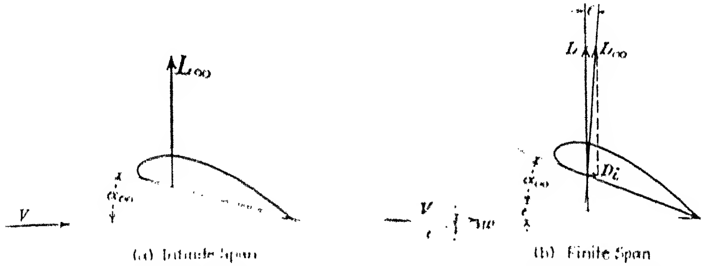


FIG. 229. Forces on an airfoil section for infinite and finite spans.

section of the vane. The total drag for a vane of finite span is then the sum of the profile and induced drags.

The calculation of the value of the induced velocity w is carried out by methods similar to those used in electricity in determining the strength of a magnetic field around a straight conductor carrying a current. The results of these calculations for a wing of elliptic plan form show that the induced drag has the value

$$D_i = \frac{2L^2}{\pi \rho b^2 V^2} \tag{53}$$

where b is the span. In terms of lift and drag coefficients, based on the plan-form area S, the result is

$$C_{Di} = \frac{C_L^2}{\pi R} \tag{54}$$

in which $R = b^2/S$ is called the aspect ratio. The induced angle may be calculated from the relation obtained from Fig. 229

$$\epsilon = \frac{D_i}{L} = \frac{C_{Di}}{C_L} = \frac{C_L}{\pi R} \tag{55}$$

These expressions are but slightly modified in the case of other plan forms. From formula (54) it may be concluded that, in order to keep the induced-drag coefficient at a minimum, the aspect ratio of an airplane wing should be made as large as possible. For a given area this means using the greatest possible span without producing a wing that is unsatisfactory from the structural standpoint. In practice values of the aspect ratio usually lie between 4 and 10.

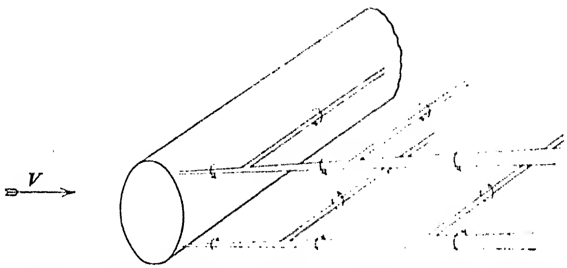


FIG. 230.—Vortex system behind a cylinder of finite length.

Although the induced-drag theory has been employed primarily in connection with airplane-design problems, its development has been made on a perfectly general basis and it may therefore be applied to the determination of the induced resistance of cylinders of the same order of fineness ratio as airfoils moving through any fluid. The problem of the airfoil has been investigated experimentally and it appears that this theory is quite satisfactory for wings of aspect ratios above 2 and for the high values of the Reynolds' numbers which are encountered in aeronautical work.

For shapes like the circular cylinder, for which the pressures are the same on the top and bottom, there is no lift and therefore no induced drag. However, when the fineness ratio is small for a symmetrical shape, or when an airfoil is set at a high angle of attack, there is evidence of a much more complicated tip flow. Thus in the case of the circular cylinder it appears that two

vortex filaments are generated at each of the ends and that these filaments join together the ends of the lateral vortices produced in the wake in the manner shown schematically in Fig. 230. There is, however, no theory analogous to that of induced drag for determining the effect of this type of flow on the resistance except the vortex-trail method developed by von Kármán for infinitely long cylinders.

Example.—An airfoil has an area of 300 sq. ft. and a span of 45 ft. and its profile is the NACA 2412 section. At what apparent angle of attack will it develop a lift coefficient of 0.5? What is the induced-drag force at this angle if the wing is attached to an airplane weighing 6500 lb. and flying horizontally at constant speed?

Solution. For infinite aspect ratio Fig. 227 shows that $C_L = 0.5$ at $\alpha = 4.75$ deg. In this case the aspect ratio is $R = b^2/S = (45)^2/300 = 6.75$ so that the induced angle, assuming that the elliptic wing formulas apply, is

$$\alpha_i = \frac{C_L}{\pi R} = \frac{0.5}{3.14 \times 6.75} = 0.0236 \text{ rad.} = 1.35^\circ$$

The apparent angle of attack is then

$$\alpha = 4.75 + 1.35 = 6.10^\circ$$

The induced drag is given by Eq. (53) and is $D_i = 2L^2/\pi\rho b^2V^2$. In level flight the lift equals the weight, that is,

$$W = L = C_L\frac{\rho V^2}{2}S$$

so that the required velocity is

$$V = \sqrt{\frac{2W}{C_L\rho S}} = \sqrt{\frac{2 \times 6500}{0.5 \times 0.002378 \times 300}} = 191 \text{ ft./sec.}$$

Then

$$D_i = \frac{2 \times (6500)^2}{3.14 \times 0.002378 \times (45)^2 \times (191)^2} = 153.2 \text{ lb.}$$

An alternative method is to find C_{D_i} and V and then to use the relation $D_i = C_{D_i}\frac{\rho V^2}{2}S$. In this case

$$C_{D_i} = \frac{C_L^2}{\pi R} = \frac{(0.5)^2}{3.14 \times 6.75} = 0.0118$$

and

$$D_i = \frac{0.0118 \times 0.002378 \times (191)^2 \times 300}{2} = 153.2 \text{ lb.}$$

Problem 348. A long airfoil having the NACA 2412 section moves through standard air at a velocity of 125 m.p.h. and at an angle of attack

of 8 deg. Determine the lift and drag forces per foot of span if the chord is 6 ft. What is the maximum lift force that this airfoil can develop at this speed?

349. Calculate the induced-drag force and coefficient for the airfoil of Prob. 348 when the span is 40 ft. and the angle of attack is 8 deg.

350. Determine the coefficients of profile, induced and total drag for the NACA 2412 airfoil when the aspect ratio has values of 4, 8 and 12 and the effective angle of attack is 12 deg.

175. Resistance of Floating Bodies.—When a body, such as the hull of a ship, floats on the surface of a liquid and at the same time is moved across this surface, a resistance to motion is produced which is somewhat more complex in nature than the resistance of a completely submerged body. As in the latter case, a portion of the drag is due to skin friction and there may also be some eddy-making drag. Besides these items there is a resistance due to the formation of waves. This new component in the expression for total resistance of the body is therefore known as the wave-making resistance.

No attempt will be made here to go into a detailed study of the nature of wave motion or of the effects of body form on wave-making resistance. In general it may be said that the formation of waves on the surface of a liquid requires a certain amount of energy and, when the waves are caused by the motion of a body, this energy is manifested in the form of an increased resistance of the body. In liquids two different types of waves may be formed. Waves of the first type, due to surface tension, are known as capillary waves or ripples and are of little importance except in the case of bodies whose dimensions are small as compared with the size of the waves. In the case of a ship, waves of the second type are more important. These are produced by the action of gravity on the masses of water which, because of differences in pressure between points on the submerged portion of the hull and the free water surface, tend to pile up around the sides of the ship. For the usual form of ship hull two sets of waves are produced, one originating at the bow and the other at the stern. Each of these sets consists of the familiar diverging waves which may be readily observed from a ship in motion; in addition there is a series of transverse waves whose crests and troughs are perpendicular to the direction of motion. The sketch shown in Fig. 231 illustrates this wave arrangement around a typical ship hull.

176. Froude's Number and Wave-making Resistance.—In order to obtain some information of an analytical character about the nature of wave-making resistance, the methods of dimensional analysis may be used to good advantage. In some cases the wave-making resistance is such a large part of the total that variations in the skin-friction drag with Reynolds' number may be neglected. Even though this is not always justifiable, the omission of viscosity and its effects from the resistance equation simplifies the analysis considerably and this factor can readily be reintroduced on the basis of the discussion of Art. 158.

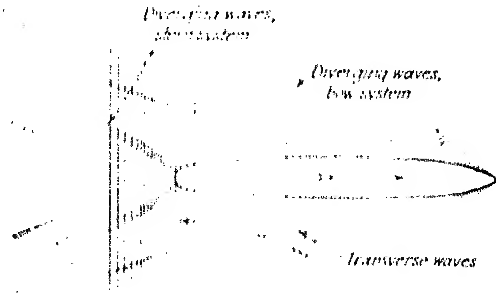


FIG. 231.—Wave pattern around a ship hull. (*J. H. Biles, "The Design and Construction of Ships," Charles Griffin and Co., Ltd., London, 1911.*)

It will therefore be assumed that the drag of a floating body in motion is dependent on the density of the fluid, ρ, the volume of displacement of the body, Δ, its velocity V and the acceleration of gravity, g, the last term appearing because of the action of gravitational attraction in producing the waves. Thus

$$D = k_1\rho^a\Delta^b V^c g^d \tag{56}$$

The introduction of the dimensions of these quantities leads to the dimensional equation

$$\frac{ML}{T^2} = \left(\frac{M}{L^3}\right)^a L^{3b}\left(\frac{L}{T}\right)^c\left(\frac{L}{T^2}\right)^d$$

The determination of the exponents by the methods used earlier in this chapter shows that three of the exponents may be expressed in terms of the fourth and that these values are

$$a = 1 \qquad b = \frac{2 + d}{3} \qquad c = 2 - 2d$$

Equation (56) may thus be written in the form

$$D = k_D \rho \Delta^{2/3} V^2 \left(\frac{\Delta^{1/3} g}{V^2} \right)^d$$

In the expression $\Delta^{1/3} g/V^2$, which carries the exponent d, the term $\Delta^{1/3}$ may be replaced by a characteristic length of the body, l, usually measured in the direction of motion. The drag then becomes

$$D = k_D' \rho \Delta^{2/3} V^2 \left(\frac{lg}{V^2} \right)^d \tag{57}$$

The combination of terms $V/\sqrt{lg}$, whose square appears in this equation, is known as Froude's number,[1] named for William Froude, the pioneer in the field of naval architecture who first introduced it. It plays a role in the expression for drag due to wave making which is analogous to that of the Reynolds' number in the resistance due to viscosity. If Froude's number is represented by the symbol N_F and the drag coefficient k_D' is replaced by $C_D/2$, then the resistance to motion is

$$D = \frac{C_D \rho V^2 \Delta^{2/3}}{2} \left(\frac{1}{N_F} \right)^{2d} \tag{58}$$

177. General Equation of Drag of a Ship Hull.—In a practical case the drag consists of viscous and eddy-making effects as well as the effect due to the formation of waves. The introduction of viscosity into the problem may again be made by the methods of dimensional analysis and it will be found that, when both viscous and gravitational forces are present, the resistance equation becomes

$$D = \frac{C_D \rho V^2 \Delta^{2/3}}{2} f_1(N_F)\, f_2(N_R) \tag{59}$$

The drag coefficient used in Eq. (59) is not the same as that in Eq. (58). A comparison of these equations shows that C_D in the former is a function of Reynolds' number and includes the term $f_2(N_R)$.

The details of the methods used in practice for separating out these various items and for converting from model test data to

[1] Some writers prefer to use the expression V^2/lg as Froude's number.

full-scale ship forms will be given in Chap. XV. The basis of Froude's method, which, with some modifications, is commonly employed for making these conversions, is that the resistance may be divided into the three separate parts: skin-friction, eddy-making and wave-making resistance. The eddy-making drag for most ship forms is quite small and variations in its magnitude within the usual range of Reynolds' numbers may be neglected. The skin-friction drag for any given Reynolds' number may be computed by using the data on flat plates and is assumed to be independent of Froude's number. The difference between the skin-friction drag and the total drag gives the "residuary resistance," in which wave- and eddy-making drags are lumped together, these quantities being considered as functions of Froude's number. In model tests, conditions of dynamic similarity between the model and its prototype are obtained when the Froude's number has the same value in both cases. Under these assumptions the resistance equation becomes

$$D = \frac{\rho V^2}{2}\left[C_f A f_1(N_R) + C_r \Delta^{2/3} f_2(N_F)\right] \tag{60}$$

where A is the "wetted surface" of the hull in contact with the water, Δ is the volume of displacement and C_r is the coefficient of residuary resistance.

Some difficulty is frequently experienced in the determination of the skin-friction drag of actual ship hulls because of the lack of exact knowledge of the condition of the surface. The surface of a new hull is quite smooth, but after a certain period of operation, depending on the climate and location, the surface becomes coated with slime and barnacles so that the roughness is a rather indeterminate quantity. As a consequence the accuracy of skin-friction drag calculations is open to question and the proper allowance to be made for these effects must be estimated by the naval architect. The methods used in different countries for the determination of the skin friction drag vary but all of them are based on the supposition that this resistance is the same as that of a flat plate parallel to the direction of motion and having a total area equal to the wetted surface of the ship, the Reynolds' number for the plate being equal to that based on the length of the hull. The methods then differ only in the relation between the skin-friction drag coefficient and the Reynolds' number. For

a full-scale ship the flow in the boundary layer is completely turbulent under the usual conditions of operation. In addition to the formulas given in Arts. 166–169 there are several empirical expressions which have been obtained by such experimenters as Froude, Gebers and Tideman.

178. Form and Resistance of Ship Hulls.—Because of the presence of wave-making resistance the ideal form for a ship hull or any floating body is quite different from that of a completely submerged body. In order to minimize the amount of energy dissipated in wave formation, it is desirable to use a pointed nose rather than a rounded one as in the case of the submerged shapes. This nose is then followed by a straight or parallel midsection and

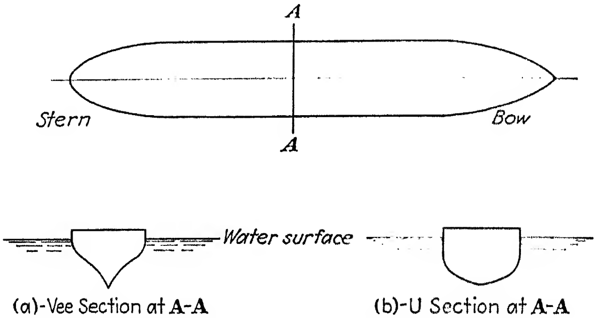


FIG. 232.—Plan and mid-sections of typical ship hulls.

the stern is rounded off, or in some slow-speed vessels is simply cut off square when there are other considerations of greater importance than a low resistance. A typical horizontal section of a ship form is as shown in Fig. 232. The vertical sections of the hull, which are also included in this figure, depend on the type of service in which the ship is to operate. For low-speed vessels the V-shaped cross section, shown in Fig. 232*a*, has been found most efficient, but for faster ships the U-shaped section of Fig. 232*b* is more satisfactory. This latter shape also has the advantage of providing more space for cargo than the V-type.

The values of the total resistance for a typical ship-hull model at different values of Froude's number, $V/\sqrt{lg}$, are shown by the curve in Fig. 233. The values of the skin-friction drag obtained by means of formula (45) for the transition range have also been determined for the corresponding values of Reynolds' number, based on the length of the hull. The difference between total

drag and skin friction gives the residuary resistance. The corresponding coefficients based on the wetted surface are also plotted in Fig. 233. It will be noticed at once that the total resistance curve has a peculiar form consisting of a series of humps and hollows with a general upward trend as the Froude's number is increased. These same humps and hollows are more prominent in the curve of total resistance coefficients. Their presence is due to the fact that at certain values of N_F the transverse waves produced at the bow and stern are superimposed so that they

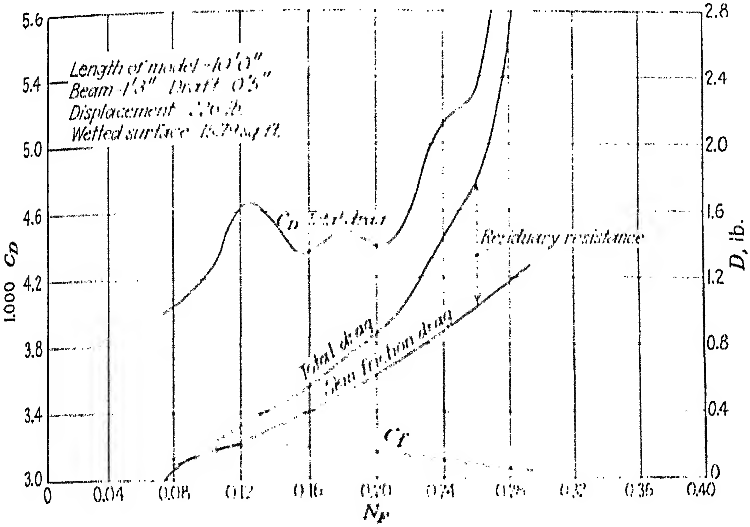


FIG. 233. Curves of total, skin friction and residuary resistance for a typical ship hull model. (*H. C. Sadler, Some Experiments on the Effect of Longitudinal Distribution of Displacement upon Resistance, Trans. Soc. Naval Arch. and Marine Eng., vol. 15, pp. 13–19, 1907.*)

are out of phase and tend to counteract each other, while at other and possibly lower values of N_F they are in phase and intensify each other so that the coefficient of wave-making resistance may actually be larger in the latter case. One of the problems in the design of a ship which is to operate at a certain speed with a given amount of power is to find the length that will give a value of N_F corresponding to one of the hollows in the curve of wave-making resistance.

In addition to the components of the total resistance of a ship which have just been discussed, there is also some drag due to the

motion of the superstructure through the air. Except for high-speed passenger vessels this drag is only a small fraction of the total and can be estimated satisfactorily. In the case of airplanes of the seaplane or flying-boat class, which are designed to land on and take off from a water surface, the resistance problem is further complicated by the fact that the hull or pontoons must operate in two different fluids. During take-off wave resistance due to motion over the surface of the water should be kept at a minimum, while after the plane has gotten into the air it is important that air resistance be kept at a minimum. It is usually impossible to find a shape which will have the minimum resistance under both sets of conditions; therefore the forms used on actual airplanes are designed on a compromise basis.

Problem 351. (*a*) A ship 325 ft. long is traveling at a speed of 25 m.p.h. What is the value of Froude's number? (*b*) A model of this ship is 12 ft. long. What must be its velocity if it is to be tested at the same value of N_F as the full-size ship?

352. A ship hull 150 ft. long has a total resistance of 6000 lb. at a speed of 22 ft. per sec. and a water temperature of 20°C. Determine the values of the residuary and skin-friction drags if the wetted surface is 3600 sq. ft. What are the values of the resistance coefficients based on the wetted surface?

General Problems

353. (*a*) An automobile has 75 hp. available for driving it forward. The rolling resistance is constant and requires 15 hp., the balance being used in overcoming air resistance. Determine the value of this resistance if the maximum speed is 80 m.p.h. in standard air. Compute the drag coefficient for wind resistance if the projected area is 32 sq. ft. (*b*) What would be the saving in power at speeds of 40 and 80 m.p.h. if the body were streamlined so that $C_D = 0.15$ Assume C_D is independent of N_R.

354. The minimum drag coefficient of an airfoil as determined from a model test in standard air at 90 m.p.h. is $C_D = 0.012$. The chord is 10 in. Estimate the value of the drag coefficient for the full-size wing when the velocity is 250 m.p.h. and the chord is 8 ft. Assume that the ratio of the full-scale and model coefficients is the same as for a flat plate on which the boundary layer is turbulent and for which the $\frac{1}{7}$-power law holds.

355. Develop expressions for the skin-friction drag of the forward- and the rear-quarter lengths of a flat plate of total length, l, placed parallel to a stream of fluid. Determine the ratio of these values. Assume that the boundary layer is completely turbulent and that its velocity distribution follows the $\frac{1}{7}$-power law.

356. (*a*) Compute the skin-friction drag for an airship hull having a maximum diameter of 35 ft., a fineness ratio of 6 and a surface area of

15,000 sq. ft. and flying at 120 ft. per sec. in standard air. Assume that this drag is equal to that of a flat plate of the same length and area and that Schlieting's formula for completely turbulent flow applies. (*b*) Calculate the drag coefficient based on the area of the maximum cross section and compare the result with the data given in Fig. 224.

357. An airplane has a weight of 5000 lb. Its wing is an NACA 2412 airfoil having an area of 250 sq. ft. and a span of 45 ft. Determine the profile, induced and total drags when it is flying in standard air at an apparent angle of attack of 4 deg. Assume lift equal to weight.

CHAPTER XIII

DYNAMICS OF COMPRESSIBLE FLUIDS

179. Elastic Properties of Fluids.—The theories concerned with the problems of fluid flow and resistance which have been discussed in the preceding chapters have all been based on the assumption that the fluid is incompressible. This assumption means that a particle of the fluid having a certain volume at, say, atmospheric pressure, will continue to have the same volume no matter how the pressure is changed. In the case of liquids such as water the hypothesis of incompressibility is not far from the truth for ordinary values of the pressure, but for gases it is obviously incorrect to assume generally that changes in pressure can take place without changes in volume. It has been shown in earlier discussions that in any fluid motion there are usually differences in pressure from one point to another, while in some cases variations in pressure at a single point may occur. If the fluid is readily compressible it is to be expected that the flow may be appreciably different from that which would exist if the fluid were incompressible. The purpose of this chapter is to consider the nature and extent of the modifications of the incompressible fluid flow which must be introduced when the effect of compressibility is considered.

The degree of compressibility of a fluid is determined by the value of its modulus of elasticity; this, as in the case of solid materials, is defined as the ratio between the stress and the corresponding strain on a particle. Fluids are never found under conditions of zero absolute pressure, that is, there is always some initial stress. In defining the modulus of elasticity, the stress is considered as an increment of pressure while the strain must correspond to the change in volume of the fluid particle produced by this pressure increment. The modulus of elasticity of a fluid is therefore frequently referred to as the bulk modulus.

If a particle of fluid having an initial volume v under a pressure p is subjected to a larger pressure $p + \Delta p$ so that the volume

decreases by an amount Δv, then the increment of stress is Δp and the increment of strain, being the change in volume per unit of volume, is $\frac{-\Delta v}{v}$. The modulus of elasticity is then

$$E = -\frac{\Delta p\, v}{\Delta v}$$

or, in the limit as Δv approaches zero,

$$E = -v\frac{dp}{dv} \tag{1}$$

Since the volume decreases when the pressure increases, dp/dv is negative so that E as given by Eq. (1) has a positive value.

For water at ordinary pressures E has a value of about 300,000 lb. per sq. in. In the case of gases it is necessary to know the relationship between pressure and specific volume in order to determine the value of this modulus. If conditions in the gas are isothermal, then

$$pv = C_1$$

and

$$\frac{dp}{dv} = -\frac{p}{v}$$

so that $E = p$. It thus appears that the modulus of elasticity of a gas is a variable and for isothermal conditions is equal to the pressure.

If the pressure-volume relation for the gas is an adiabatic one, then

$$pv^k = C_2$$

and

$$\frac{dp}{dv} = -\frac{kp}{v}$$

In this case $E = kp$.

Problem 358. A closed cylinder 6 in. in diameter contains 3.5 cu ft. of water at an average pressure of 16 lb. per sq. in. abs. What is the decrease in its volume if the pressure is increased to 20 lb. per sq. in. abs.? If one end of the cylinder is moved inwardly a distance of 0.75 in., what is the pressure?

359. Air at a pressure of 15 lb. per sq. in. abs. and a temperature of 120°F. is allowed to expand isothermally until the pressure is 20 lb. per sq. in. abs. Determine the initial and final values of the bulk modulus and the specific volume.

360. Solve Prob. 359 for the case of an adiabatic expansion.

361. Determine the values of the bulk modulus for air and methane at atmospheric pressure for isothermal and adiabatic conditions.

180. Pressure Waves. The Velocity of Sound.—When a disturbance producing a change in pressure occurs at a point in a fluid, this disturbance is propagated through the fluid in the form of a longitudinal wave of the same type as a sound wave. Such a wave consists of a series of condensations and rarefactions which move through the fluid with a certain velocity that is superimposed on any other motion which the fluid may have. It will be shown presently that the velocity of the wave is independent of the wave length and will therefore have the same value for any longitudinal wave no matter what the cause of the disturbance.

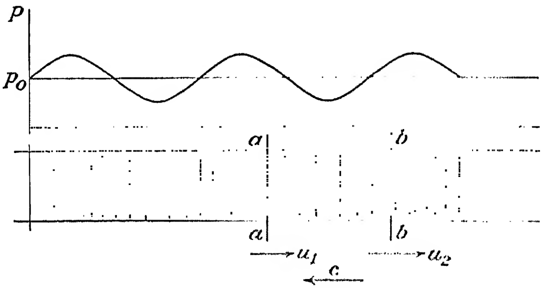


FIG. 234.—Longitudinal wave motion in a straight tube.

The velocity of propagation of a wave form is identical with the velocity of sound through the fluid, since sound is known to be transmitted by waves of this type which merely happen to have frequencies that make them audible.

In order to determine the velocity of propagation of a longitudinal wave, it may be supposed that such a wave is traveling through a tube of constant cross section, as shown in Fig. 234, the fluid in the tube being assumed to be initially at rest. The positions of the condensations and rarefactions at a particular instant are shown by the vertical lines drawn across the tube while the variation in pressure along the tube at this instant is given by the graph. It is assumed that a continuous series of waves is passing through the fluid from right to left. For the purpose of analysis a velocity equal but opposite to that of the wave motion is imposed on the fluid in the tube. Then with respect to the tube the wave form is at rest, but the fluid particles

have velocities which are the vector sums of the velocity just added and those of the oscillatory motion of the wave.

The condition of continuity is now applied to the fluid contained between two sections of the tube *aa* and *bb*. Since the wave has been fixed in position, the velocity due to the wave motion alone, the pressure and the density are constant for any one section of the tube. The values of these quantities will be designated by p_1, ρ_1 and u_1 at section *aa* and p_2, ρ_2 and u_2 at section *bb*, and the velocity of propagation will be represented by c. The mass of fluid contained between the sections *aa* and *bb* is constant so that the amount entering through *bb* must equal that leaving through *aa*. This mass is therefore

$$m = (c - u_1)\rho_1 A = (c - u_2)\rho_2 A \tag{2}$$

where A is the cross sectional area of the tube.

Now the force acting on the mass of fluid due to the difference in pressure on its ends is

$$(p_2 - p_1)A \tag{3}$$

and this force must be equal to the change in momentum of the fluid in unit time. The momentum of the fluid mass m at section *bb* is

$$M_{bb} = m(c - u_2)$$

while that at section *aa* is

$$M_{aa} = m(c - u_1)$$

so that the net change in momentum in unit time from *aa* to *bb* is

$$\Delta M = M_{aa} - M_{bb} = m(u_2 - u_1)$$

The values of u_1 and u_2 may be obtained from Eq. (2) and are

$$u_1 = c - \frac{m}{A\rho_1} \qquad u_2 = c - \frac{m}{A\rho_2}$$

When these values are substituted in the expression for the change in momentum, the result is

$$\Delta M = \frac{m^2}{A}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$$

or, in terms of the specific volumes v_1 and v_2 and the acceleration of gravity,

$$\Delta M = g\frac{m}{A}(v_1 - v_2) \tag{4}$$

The change in momentum in unit time represented by Eq. (4) is equal to the pressure force as given by Eq. (3), that is,

$$(p_2 - p_1)A = g\frac{m^2}{A}(v_1 - v_2)$$

Solving for the ratio of the mass of fluid to the cross-sectional area of the tube, the result is

$$\frac{m^2}{A^2} = \frac{p_2 - p_1}{g(v_1 - v_2)} \tag{5}$$

The modulus of elasticity may be expressed in terms of the pressures and specific volumes used above. The increment of stress on a particle moving from *bb* to *aa* is now equal to the change in pressure between the two sections of the tube, that is, $p_1 - p_2$, while the corresponding strain is $(v_1 - v_2)/v_1$. Thus the modulus has the value

$$E = -\left(\frac{p_1 - p_2}{v_1 - v_2}\right)v_1 = \left(\frac{p_2 - p_1}{v_1 - v_2}\right)v_1$$

When this value is substituted in Eq. (5), the result obtained is

$$\frac{m^2}{A^2} = \frac{E}{gv_1} \tag{6}$$

The quantity of fluid affected by the wave motion may be calculated in a slightly different manner if it is assumed that the changes in pressure, density and velocity due to the wave motion alone are small in comparison with their average values. If these average values are assumed to be equal to the values they would have if there were no wave motion, but only a uniform flow through the tube of velocity c, then the mass passing any cross section is

$$m = c\rho A = \frac{cA}{vg}$$

so that

$$\frac{m}{A} = \frac{c}{vg} \tag{7}$$

where ρ and v are the density and specific volume, respectively, of the fluid before any change of pressure due to the wave. In

Eq. (6) the specific volume v_1 may be considered as equal to v so that, if the values of m^2/A^2 obtained from Eqs. (6) and (7) are equated, the result may be solved for the velocity of propagation, c, which turns out to be

$$c = \sqrt{Evg} = \sqrt{\frac{E}{\rho}} \tag{8}$$

This result was first obtained by Newton and the proof given above is essentially that presented in advanced textbooks in physics.[1]

The definition of the modulus of elasticity as given by Eq. (1) was

$$E = -v\frac{dp}{dv}$$

Now

$$\frac{dp}{dv} = \left(\frac{dp}{d\rho}\right)\left(\frac{d\rho}{dv}\right)$$

and since $v = 1/\rho g$, then

$$\frac{d\rho}{dv} = -\rho^2 g$$

so that

$$E = \rho\frac{dp}{d\rho}$$

The expression (8) for the velocity of propagation then becomes

$$c = \sqrt{\frac{dp}{d\rho}} \tag{9}$$

The velocity of propagation for a longitudinal wave is also the velocity of sound in the medium. In the case of air, if no change in temperature occurs as the wave passes through it, the pressure and density are related by the expression

$$\frac{p}{\rho} = C_1'$$

the right-hand term being a constant. Then

$$\frac{dp}{d\rho} = C_1' = \frac{p}{\rho}$$

[1] See, for example, W. WATSON, "A Text Book of Physics," 7th ed., pp. 364–367, Longmans, Green & Co., London, 1920.

so that

$$c = \sqrt{\frac{p}{\rho}} \tag{10}$$

For air under standard atmospheric conditions at a temperature of 59°F., $p = 2116.8$ lb. per sq. ft. while $\rho = 0.002378$ slugs per cu. ft. and the velocity of sound has the value

$$c = 943.5 \text{ ft./sec.}$$

This value is considerably lower than the experimentally determined velocity of 1120 ft. per sec.

The explanation of this large discrepancy lies in the fact that the temperature does not actually remain constant. The changes in pressure are usually so rapid that there is no opportunity for heat to be transferred to or from the fluid or from one particle of fluid to another. The relationship between pressure and density is therefore more accurately represented by the adiabatic law

$$\frac{p}{\rho^k} = C_2'$$

where C_2' is a constant and k is the ratio of specific heats at constant pressure and constant volume. In this case

$$\frac{dp}{d\rho} = kC_2'\rho^{k-1} = \frac{kp}{\rho}$$

and the velocity of sound is

$$c = \sqrt{\frac{kp}{\rho}} \tag{11}$$

For standard air, taking $k = 1.406$, this formula yields the value of 1118.7 ft. per sec., which is in excellent agreement with the experimental value. In fact Eq. (11) is considered by physicists to be sufficiently exact to serve as the basis for the experimental determination of k for various gases. The velocity of sound is an important factor in all studies of the motion of compressible fluids. It is often referred to as the acoustic velocity.

For air at any other conditions of pressure and density the ratio of pressure to density may be obtained from the gas law $pv = RT$. Putting $v = 1/\rho g$, this gives $p/\rho = gRT$ so that the velocity of sound is

$$c = \sqrt{kgRT} \tag{12}$$

Equation (12) shows that for a given gas the velocity of sound varies only with the absolute temperature, the other terms under the radical being constant.

Problem 362. Determine the velocity of sound in water and in methane, the latter having a density of 0.00132 slugs per cu. ft. at a pressure of 14.7 lb. per sq. in. abs.

363. Plot a curve showing the acoustic velocity in air for temperatures from 0 to 150°F.

364. What is the value of k for a gas if it is found experimentally that the acoustic velocity is 1550 ft. per sec. when the pressure is 20 lb. per sq. in. abs. and the density is 0.0015 slugs per cu. ft.?

181. Bernoulli's Theorem for Compressible Nonviscous Fluids.

In discussing the dynamic properties of the motion of a compressible fluid, considerable light may be shed on the subject by a reconsideration of Bernoulli's theorem. The assumption of constant density will now be abandoned but it will be supposed that some analytical relationship between the pressure and density still exists. The condition of equilibrium of a particle of fluid, as given by Eq. (14), page 83, under the assumption that viscosity may be neglected is

$$dp + \rho g\, dz + \frac{\rho}{2} d(V^2) = 0$$

or

$$\frac{dp}{\rho} + g\, dz + \frac{1}{2} d(V^2) = 0 \qquad (13)$$

If the average motion is assumed to be steady and streamline in character, then this equation may be integrated along a streamline, the result being

$$\int \frac{dp}{\rho} + gz + \frac{V^2}{2} = E' \qquad (14)$$

in which E' is a constant for any one streamline and is now the total energy per unit of mass.

In order to evaluate the integral which constitutes the first term of Eq. (14), a relationship between the pressure and density must be introduced. The physical nature of the motion of the fluid is again such that the assumption of an adiabatic law is a rational one. The pressure and density are therefore related by the formula $p/\rho^k = C_2'$ so that the pressure-density integral of Eq. (14) becomes

$$\int \frac{dp}{\rho} = C_2'k \int \rho^{k-2}\, d\rho = \frac{C_2'k}{k-1}\rho^{k-1} = \frac{k}{k-1}\frac{p}{\rho}$$

and Bernoulli's theorem is then

$$\frac{k}{k-1}\frac{p}{\rho} + gz + \frac{V^2}{2} = E' \tag{15}$$

In most cases the term due to the elevation of the particle of fluid may be neglected and Bernoulli's theorem then becomes

$$\frac{k}{k-1}\frac{p}{\rho} + \frac{V^2}{2} = E' \tag{16}$$

The constant E' on the right side of Eq. (16) may be eliminated by considering the pressures, densities and velocities at two points on the same streamline which are designated by the numbers 1 and 2. Writing Bernoulli's theorem for these two points leads to the result

$$V_2^2 - V_1^2 + \frac{k}{k-1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) = 0$$

The relation between the pressures and densities at the two points is

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^k$$

and the introduction of this expression into the second term of the above equation makes it possible to rewrite the latter in the form

$$V_2^2 - V_1^2 + \frac{k}{k-1}\frac{p_1}{\rho_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right] = 0 \tag{17}$$

Referring to Eq. (11) it appears that the combination of terms kp_1/ρ_1 in Eq. (17) is equal to the square of the velocity of sound at the point designated by the subscript 1. This velocity will be found to be of great significance in dealing with the flow of compressible fluids.

Substituting $c_1^2 = kp_1/\rho_1$ in Eq. (17), the expression becomes

$$V_2^2 - V_1^2 + \frac{c_1^2}{k-1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right] = 0 \tag{18}$$

The ratio of the pressures at the two points in question may be calculated from Eq. (18) and is found to be

$$\frac{p_2}{p_1} = \left[1 - \frac{k-1}{2k}\,\frac{\rho_1}{p_1}\left(V_2^2 - V_1^2\right)\right]^{\frac{k}{k-1}} \tag{19}$$

182. Pressure at a Stagnation Point. An interesting application of this analysis is found in the steady flow of a uniform stream of gas past a solid body. The velocity, pressure and density at a point in the undisturbed stream corresponding to point 1 in Eq. (19) may be represented by V_0, p_0 and ρ_0, respectively, while at the stagnation point corresponding to point 2 the pressure is p_s, the density is ρ_s and the velocity is $V_s = 0$. Making these substitutions, Eq. (19) becomes

$$\frac{p_s}{p_0} = \left[1 + \frac{k-1}{2}\,\frac{V_0^2}{c_0^2}\right]^{\frac{k}{k-1}} \tag{20}$$

where c_0 is the acoustic velocity in the undisturbed fluid.

If the ratio V_0/c_0 is small enough so that the term $\frac{k-1}{2}\frac{V_0^2}{c_0^2}$ is less than unity, then the right side of the above expression may be expanded into a convergent series by means of the binomial theorem. The result is

$$\frac{p_s}{p_0} = 1 + \frac{kV_0^2}{2c_0^2}\left[1 + \frac{V_0^2}{4c_0^2} - \frac{(k-2)}{24}\,\frac{V_0^4}{c_0^4} + \cdots\right]$$

If both sides of this expression are multiplied by p_0 and the quantity kp_0/c_0^2 is replaced by its equivalent ρ_0, then the formula for pressure at the stagnation point becomes

$$p_s = p_0 + \frac{\rho_0 V_0^2}{2}\left[1 + \frac{V_0^2}{4c_0^2} - \frac{(k-2)}{24}\,\frac{V_0^4}{c_0^4} + \cdots\right] \tag{21}$$

In the case of an incompressible fluid the pressure at a stagnation point was found to be [see Eq. (18), page 87]

$$p_s = p_0 + \frac{\rho_0 V_0^2}{2} \tag{22}$$

A comparison of Eqs. (21) and (22) shows that compressibility causes an increase in the stagnation-point pressure, this increase

being dependent on the ratio of the velocity of flow to the acoustic velocity in the undisturbed stream.

The ratio V/c will be found to appear in all flow problems where compressibility is an important factor. It is known as Mach's number, being so named in honor of the Austrian scientist who was responsible for much of the early work in this field. In what follows Mach's number will be represented by the symbol N_M. The ratio V_0/c_0 in Eq. (21) may be replaced by N_M so that the pressure at a stagnation point is then

$$p_s = p_0 + \rho_0 V_0^2 \quad 1 + \frac{1}{4}N_M^2 - \frac{(k-2)}{24}N_M^4 \cdots \Big] \tag{23}$$

When N_M is less than unity the series is convergent and when N_M is small the error caused by assuming the fluid to be compressible is not large. Thus, in the case of air at standard atmospheric conditions moving with a velocity of 280 ft. per sec., $N_M = 280/1120 = 1/4$ and the stagnation-point pressure is

$$p_s = p_0 + \qquad \Big) = p_0 + 1.0157\frac{\rho_0 V_0^2}{2}$$

so that at this speed the error in the measurement of the pressure difference $p_s - p_0$ is only 1.57 per cent.

The stagnation-point pressure in a stream of gas may be determined satisfactorily by means of Eqs. (20) and (23) as long as the velocity of the flow does not exceed that of sound or as long as Mach's number does not become greater than unity. For speeds in the range above $N_M = 1$, the assumption of adiabatic changes is no longer sufficiently accurate and a more elaborate theoretical treatment of the problem must be employed.[1]

The theory developed above has an important application in connection with the use of such instruments as the pitot-static tube for the measurement of velocity in gases. It is apparent from Eqs. (21) and (22) that the pressure difference $p_s - p_0$ is larger in a compressible fluid than in an incompressible one for the same speed of the stream. If the measured value of this pressure difference is used in the formula for velocity [Eq. (20), page 87], based on the assumption of an incompressible medium,

[1] LORD RAYLEIGH, "Scientific Papers," vol. VI, pp. 407–415, Cambridge University Press, 1920.

the velocity obtained will be higher than the true value by an amount that increases with Mach's number. The values of the stagnation-point pressures in air with and without the assumption of compressibility have been computed by Zahm[1] for a large range of velocities, the variation of this pressure with velocity being shown graphically in Fig. 235.

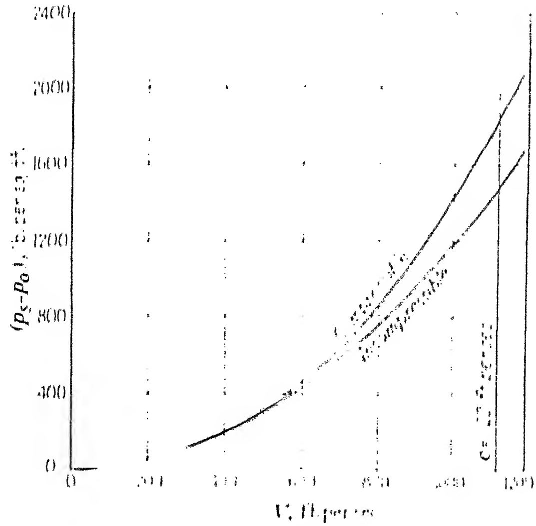


FIG. 235.—Variation of stagnation-point pressures with speed for air with and without the assumption of compressibility.

Problem 365. In the undisturbed portion of a stream of air which flows past a body, the pressure is 14.7 lb. per sq. in. abs., $\rho = 0.002378$ slugs per cu. ft., the velocity is 500 m.p.h. and the acoustic velocity is 1120 ft. per sec. What is the velocity at a point near the body where the pressure is 13.9 lb. per sq. in. abs.? What would be the velocity at this point if the fluid were assumed incompressible?

366. The velocity at a point on a body immersed in a stream of air is 720 ft. per sec. In the undisturbed stream the velocity of flow is 540 ft. per sec., the pressure is 14.5 lb. per sq. in. abs. and $\rho = 0.0021$ slugs per cu. ft. What is the pressure at the point on the body?

367. At point A in an airstream the velocity is 320 ft. per sec., the pressure is 14.2 lb. per sq. in. abs. and the density is 0.00229 slugs per cu. ft. At another point B the pressure is 13.4 lb. per sq. in. abs. Determine the value of Mach's number for both of these points.

[1] ZAHM, A. F., Pressure of Air on Coming to Rest from Various Speeds, *NACA Tech. Rept.* 247, 1926.

368. What is the stagnation-point pressure in a stream of carbon dioxide if in the undisturbed stream the pressure is 20.3 lb. per sq. in. abs., the density is 0.0042 slugs per cu. ft. and the velocity is 425 ft. per sec.?

183. Stream Tubes in a Compressible Fluid.—The determination of the effect of compressibility at points in a moving fluid other than a stagnation point is rather difficult if Eq. (16) is used alone. A method which, though qualitative, is much simpler and gives more easily interpreted information is based on an analysis of the condition of continuity as applied to a stream tube in the fluid. If the cross-sectional area of such a tube is represented by A and the density and velocity at any point in it are represented by ρ and V, respectively, the condition of continuity for the tube is

$$\rho VA = \text{constant}$$

It is desired to study the manner in which the area A varies with the velocity V. Differentiating with respect to V and then dividing by ρVA, the result obtained is

$$\frac{1}{A}\frac{dA}{dV} + \frac{1}{V} + \frac{1}{\rho}\frac{d\rho}{dV} = 0 \tag{24}$$

Bernoulli's theorem in the form of Eq. (16) may also be differentiated with respect to V, giving the expression

$$V + \frac{k}{k-1}\left(\frac{1}{\rho}\frac{dp}{dV} - \frac{p}{\rho^2}\frac{d\rho}{dV}\right) = 0$$

or

$$V + \frac{k}{k-1}\frac{d\rho}{\rho\, dV}\left(\frac{dp}{d\rho} - \frac{p}{\rho}\right) = 0 \tag{25}$$

Since the adiabatic relation between pressure and density is assumed to hold, the acoustic velocity may be written in either of the two forms

$$c = \sqrt{\frac{dp}{d\rho}} \qquad \text{or} \qquad c = \sqrt{\frac{kp}{\rho}}$$

and the introduction of these values in Eq. (25) gives

$$V + \frac{c^2}{\rho}\frac{d\rho}{dV} = 0$$

so that

$$\frac{d\rho}{dV} = -\frac{\rho V}{c^2}$$

The substitution of this quantity in Eq. (24) gives for the rate of change of area with velocity the expression

$$\frac{dA}{dV} = -\frac{A}{V}\left(1 - \frac{V^2}{c^2}\right) = -\frac{A}{V}(1 - N_M{}^2) \tag{26}$$

This result may be interpreted by considering the two cases of flow in which the velocity is either above or below the acoustic velocity and in which Mach's number is therefore either greater or less than one. When $N_M < 1$ the value of dA/dV is negative, indicating that an increase in the velocity causes the stream tube to diminish in cross-sectional area but at a slower rate than in an incompressible fluid. When the velocity of the fluid exceeds that of sound and $N_M > 1$, then dA/dV is positive and the stream tubes increase in size when the velocity becomes still larger. It thus appears that there is a decided difference in the type of flow that takes place, depending on whether the velocity is above or below the acoustic velocity.

184. The Venturi Meter for Compressible Fluids.—The theory of the Venturi meter for incompressible fluids has been given in Arts. 50 and 51 of Chap. V and the formulas worked out there are satisfactory for liquids, but when the fluid is a gas the effect of compressibility must be considered. Applying Eq. (18) to the inlet and throat sections, denoted by the subscripts 1 and 2, respectively, the following relationship between the corresponding velocities is obtained

$$V_2{}^2 - V_1{}^2 = \frac{2c_1{}^2}{k-1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right] \tag{27}$$

The equation of continuity is now $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$ so that the velocity at the throat is

$$V_2 = \frac{\rho_1}{\rho_2}\frac{A_1}{A_2}V_1 \tag{28}$$

Again assuming an adiabatic relation between pressure and density, the ratio of densities is found to be

$$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right) \tag{29}$$

If the area ratio $A_1/A_2 = n$, then, substituting this value and Eq. (29) in Eq. (28), the throat velocity is

$$V_2 = \frac{nV_1}{(p_2/p_1)^{\frac{1}{k}}}$$

and the expression for velocity at the inlet obtained from Eq. (27) is

$$V_1 = \frac{c_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{k}}\left\{\frac{2}{k-1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]\right\}^{\frac{1}{2}}}{n^2-\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}}} \tag{30}$$

It is convenient to express the velocity in terms of the temperature rather than the density of the gas. The value of the velocity of sound at the inlet is

$$c_1 = \sqrt{\frac{kp_1}{\rho_1}}$$

and, since the gas law $p/g\rho = RT$ is applicable, the ratio of pressure to density at the inlet may be written as

$$\frac{p_1}{\rho_1} = \frac{p_0}{\rho_0}\frac{T_1}{T_0}$$

where the subscript 0 refers to some standard condition of temperature and pressure for which the density ρ_0 is known. The acoustic velocity in terms of the temperature is

$$c_1 = \sqrt{\frac{kp_0}{\rho_0}\frac{T_1}{T_0}}$$

and, substituting this value in Eq. (30), the expression for the inlet velocity is found to be

$$V_1 = \frac{\left\{\frac{2k}{k-1}\frac{p_0}{\rho_0}\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]\right\}^{\frac{1}{2}}}{\left[n^2-\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}}\right]^{\frac{1}{2}}}\sqrt{\frac{T_1}{T_0}} \tag{31}$$

If the coefficient of the temperature term $\sqrt{T_1/T_0}$ is represented by Y then the velocity is equal to

$$V_1 = Y\sqrt{\frac{T_1}{T_0}} \tag{32}$$

Values of Y for air under the conditions of 760 mm. mercury pressure and a temperature of 0°C. have been calculated[1] for several values of n and values of p_2/p_1 ranging from 1 to 0.6. These values are given in Table XI. In general a Venturi meter should be calibrated against some other method of measuring velocity or discharge, but the theoretical formulas serve to indicate the nature of the results that may be expected.

TABLE XI. VALUES OF Y, FT. PER SEC.

p_2/p_1	$n = 4$	$n = 9$	$n = 16$
0.9998	4.74	2.05	1.150
0.999	10.60	4.59	2.57
0.995	23.65	10.24	5.74
0.99	33.34	14.11	8.09
0.98	46.48	20.3	11.38
0.95	72.8	31.6	17.7
0.90	99.8	43.4	24.3
0.80	131.7	57.5	32.2
0.60	157.9	69.3	38.9

Computed on the assumptions $pv = RT$, c_v = constant; $c_p/c_v = 1.400$
$p_0 = 14.692$ lb./sq. in.
$\rho_0 = 0.0012928$ g./cm.^{3} at 760 mm. and 0°C.

It is found that the theory of the Venturi meter based on the assumption of an incompressible fluid gives satisfactory results for a gas only when the pressure ratio p_2/p_1 is close to unity. The inlet velocity in the case of an incompressible fluid as given by Eq. (30), page 97, is

$$V_1 = K\sqrt{\frac{2(p_1 - p_2)}{\rho}}$$

For purposes of comparison with Eq. (31), the pressure ratio p_2/p_1 may be introduced. Putting $K = \sqrt{1/(n^2 - 1)}$, the expression for the inlet velocity is

[1] BUCKINGHAM, E., The Theory of the Pitot and Venturi Tubes, *NACA Tech. Rept.*, vol. 1, no. 2, p. 110, 1916.

$$V_1 = \sqrt{\frac{2p_1}{(n^2 - 1)\rho}\left(1 - \frac{p_2}{p_1}\right)} \tag{33}$$

An inspection of Eqs. (31) and (33) shows that for $p_2/p_1 = 1$, $V_1 = 0$ in both cases. For other values of p_2/p_1 a comparison is difficult except by consideration of a numerical example. Suppose that a Venturi tube has an area ratio of 4 and is placed in an air line. The pressure and temperature at the Venturi inlet are 14.7 lb. per sq. in. and 59°F., respectively, so that

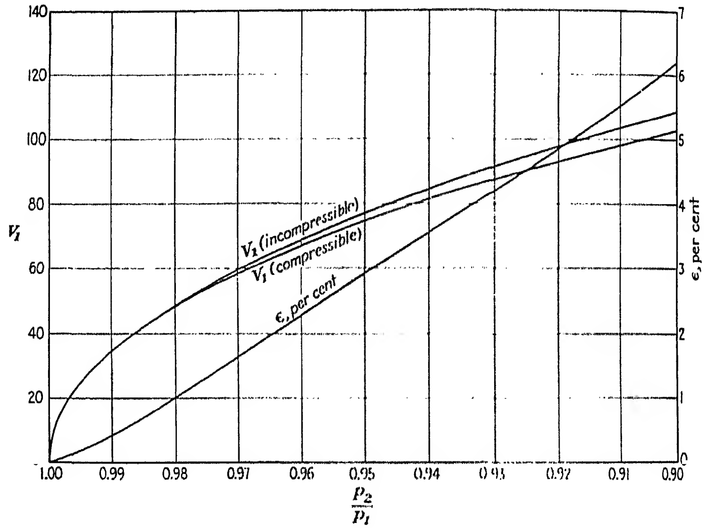


FIG. 236.—Effect of compressibility on Venturi tube characteristics.

$c_1 = 1120$ ft. per sec. The value of k is taken as 1.406 and the velocities are computed for values of p_2/p_1 running from 1.0 to 0.9, using formulas (30) and (33), the results being plotted graphically in Fig. 236. It appears from the figure that the error in using the incompressible fluid formula is less than, say, 3 per cent only when the pressure ratio is greater than 0.951.

When the same meter is used in a water line, formula (33) gives satisfactory results for a much larger range of pressure ratios; in fact conditions seldom arise in practice where this formula is not suitable for use with liquids. For gases it is in general difficult and unsafe to draw any exact conclusions about the range of pressures in which compressibility may be neglected except

by consideration of specific cases, as in the above example. Although formulas (30) and (31) are considerably more complicated than Eq. (33), the use of the first two is advisable for all gases while the latter expression should be employed only with liquids.

Problem 369. A Venturi meter having an area ratio of 3 is installed in a 6-in. air line. The pressures at the inlet and throat are 145 lb. per sq. in. abs. and 133 lb. per sq. in. abs., respectively, and the temperature at the inlet is 80°F. What are the velocity at the inlet and the weight discharge?

370. A Venturi meter having an area ratio of 2.5 is placed in a pipe line carrying acetylene. At the inlet the pressure is 110 lb. per sq. in. abs. and the temperature is 75°F., while at the throat the pressure is (*a*) 105 lb. per sq. in. abs. (*b*) 96.8 lb. per sq. in. abs. Determine the inlet velocity in each case, assuming the gas to be incompressible and then compressible. For acetylene $R = 59.34$ ft. per °F.

185. Resistance in Compressible Fluids.—There are a number of important problems in which velocities greater than that of sound may be attained. For example, these may be cases where solid bodies move through a large expanse of fluid at high speeds that exceed the acoustic velocity. Examples are found in the flight of rifle bullets and the motion of the elements of high-speed airplanes. The problem of determining the effect of compressibility on the resistance of such objects at speeds both above and below the acoustic velocity will now be discussed in detail.

As in the discussion of viscous and wave resistance, the methods of dimensional analysis may be employed advantageously in obtaining a general resistance equation for high-speed motion. It is now assumed that the resistance depends on the density of the fluid, the projected area of the body and its velocity, as before, and in addition on the rate at which pressures are propagated through the fluid, in other words, on the velocity of sound. Thus the formula for the drag of a body becomes

$$D = k_D \rho^a A^b V^d c^e \tag{34}$$

in which a, b, d, and e are unknown exponents. The introduction of the dimensions of the various quantities involved in this equation leads to the expression

$$M\frac{L}{T^2} \approx \left(\frac{M}{L^3}\right)^a L^{2b}\left(\frac{L}{T}\right)^d\left(\frac{L}{T}\right)^e$$

On equating separately the exponents of mass, length and time for the two sides of the equation, three simultaneous equations are obtained which are as follows:

$$\begin{aligned} 1 &= a \\ 1 &= -3a + 2b + d + e \\ -2 &= -d - e \end{aligned}$$

The solution of these equations gives for the exponents of Eq. (34) the values

$$\begin{aligned} a &= 1 \\ d &= 2 - e \\ b &= 1 \end{aligned}$$

so that the value of the drag is then

$$D = k_D \rho A V^{2-e} c^e = k_D \rho A V^2 \left(\frac{c}{V}\right)^e \tag{35}$$

or, in terms of the dynamic pressure and Mach's number,

$$D = C_D \frac{\rho V^2}{2} A \left(\frac{1}{N_M}\right) \tag{36}$$

It thus appears that Mach's number plays an important part in determining the resistance of a body moving through a compressible fluid.

186. Motion at Subsonic and Supersonic Velocities.—The dimensional analysis used in deriving Eq. (36) does not make any distinction between flows at velocities above and below the velocity of sound. Only the investigation of Art. 183 on the behavior of stream tubes is available as an indication that the value of $N_M = 1$ forms a demarcation between two radically different types of flow with correspondingly different values for the function of Mach's number that appears in Eq. (36). Some further insight into this problem may be obtained by considering the cases of an infinitesimal particle or point disturbance moving through a fluid, first at a subsonic and then at a supersonic velocity. The first case is illustrated in Fig. 237. Point A represents the initial position of the particle assumed to be traveling along the line AB with a uniform velocity V which is less than the acoustic velocity c. At the instant the particle passes point A, a pressure wave having a spherical front with point A as its

center is produced. This wave travels outward in all directions with the velocity c so that, if the particle reaches point B in time t, the radius of the wave front will then be ct, while the distance $AB = Vt$. The intermediate points between A and B are sources of other spherical pressure waves but, since the initial wave starting from A is moving faster than the particle, these will always be contained within the sphere of radius ct.

When the particle moves at a supersonic velocity, the situation is as illustrated in Fig. 238. As in the previous case a spherical

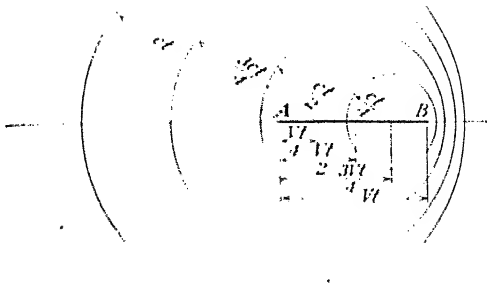


FIG. 237. Wave front produced by a particle moving at subsonic velocity.

pressure wave originates at A at the instant the particle passes that point, but in the time t required to reach point B the particle has traveled a greater distance than the radius of the sphere whose center is at A. Spherical waves also emanate from the intermediate points between A and B and have radii which are proportional to the distances of these points from B. Thus the entire system of spherical pressure waves combines to form a conical front with its vertex at B. The half angle α at the vertex of this cone is readily shown to be equal to

$$\alpha = \sin^{-1} \frac{ct}{Vt} = \sin^{-1} \frac{c}{V} = \sin^{-1} \frac{1}{N_M} \qquad ($$

The angle α is commonly known as Mach's angle.

The fundamental difference between the two types of m[illegible] may perhaps be clarified by the following discussion. In th[illegible] of subsonic velocities the particle or body is able to tele

ahead, by means of the pressure waves which it sets up, the fact that it is approaching so that the upstream particles of the fluid have an opportunity to adjust themselves at least partially to the motion before the body reaches their positions. On the other hand, when the body moves at a supersonic velocity these pressure waves fall behind so that, as the body comes in contact with the fluid particles which were initially ahead of it, a considerable shock is involved. For this reason the conical wave front shown

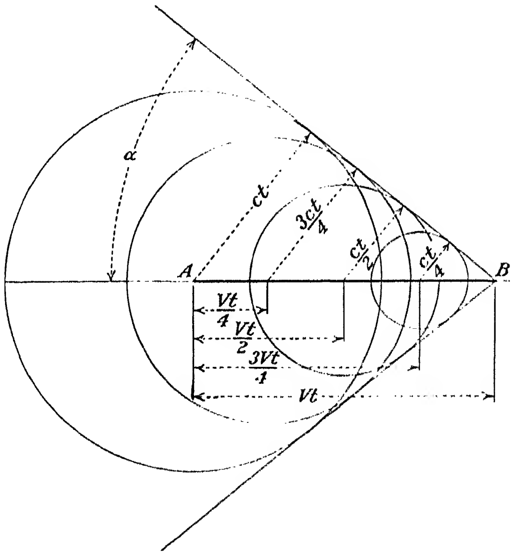


FIG. 238.—Wave front produced by a particle moving at supersonic velocity.

in Fig. 238 is known as a shock wave. Actually the shock wave is a form of discontinuity and in real fluids, such as air, has a finite thickness. Conditions within the shock wave have been the subject of considerable study[1] but this problem is too advanced for treatment here.

187. Effects of Compressibility on Resistance.—Although the discussion of the preceding article was based on a study of the motion of an infinitesimal solid particle, the conclusions reached are in general applicable to bodies of finite dimensions. It would thus seem that there would be a considerable difference in the

[1] TAYLOR, G. I., and J. W. MACCOLL, "The Mechanics of Compressible Fluids," Div. H, vol. II, of "Aerodynamic Theory," edited by W. F. Durand, Julius Springer, Berlin, 1935.

values of the resistance coefficient of a body for velocities above and below the acoustic velocity. Experimental verification of this supposition is shown in Fig. 239,[1] in which the drag coefficients for two artillery projectiles as obtained from the equation

$$D = C_D \frac{\rho V^2}{2} A$$

are plotted against Mach's number. In this equation the coefficient C_D is equivalent to $C_D(1/N_M)^e$ of Eq. (36). The curves show that there is a sudden increase in the value of C_D in the

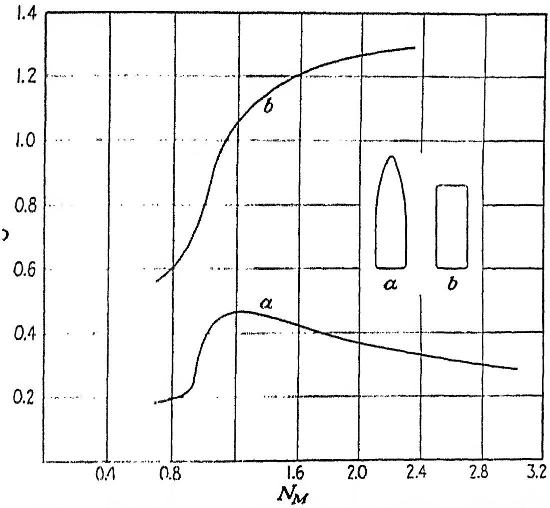


Fig. 239. Variation of C_D with N_M for artillery projectiles.

neighborhood of $N_M = 1$. It is also interesting to note how the sharp nose of projectile a causes an appreciable drop in the value of C_D beyond $N_M = 1.2$, although it never becomes so low as the values for speeds well below that of sound. The higher drag of the cylindrical projectile is due to the fact that its blunt nose produces a stronger shock wave than that of the sharp-nosed shell.

A photographic method available for the study of the external form of shock waves was originated by Töpler[2] and has yielded

[1] Figures 239, 240 and 241 are reprinted from the article by J. Ackeret, "Gasdynamik," in "Handbuch der Physik," vol. VII, pp. 336–338, Julius Springer, Berlin, 1927.

[2] Ewald, P. P., T. Pöschl and L. Prandtl, "The Physics of Solids and Fluids," p. 261, Blackie & Son, Ltd., London, 1930.

some extremely valuable results. The procedure, known as the *schlieren* method, consists in passing light through the region of the gas in motion. Because the refraction of light by the gas depends on the density of the latter, the flow when illuminated

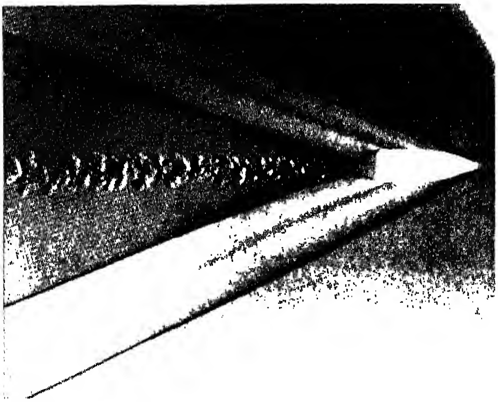


FIG. 240.—*Schlieren* photograph of flow produced by a sharp-nosed projectile moving at a supersonic velocity.

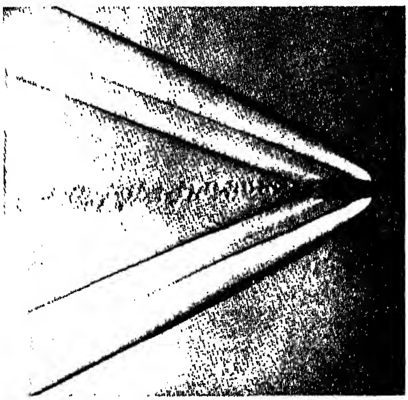


FIG. 241.—*Schlieren* photograph of flow produced by a blunt-nosed projectile moving at a supersonic velocity.

in this manner and photographed gives a picture in which the shock waves are represented by sharp, dark lines. Photographs of this kind are shown in Figs. 240 and 241 for sharp- and blunt-nosed projectiles similar in shape to those for which the drag coefficients are given in Fig. 239.

In the case of the sharp-nosed shell the forward shock wave appears to be essentially a straight line running out from the tip so that the value of Mach's angle can be readily determined. Such measurements, along with the use of Eq. (37), make it possible to determine the value of Mach's number for the motion and consequently the speed of the projectile. The shock wave formed at the nose of the blunt shell is not so sharp nor is its contour a straight line but, if the measurements of Mach's angle are based on the straight portions of the wave behind the shell, fairly accurate speed determinations can be made.

The resistance of a high-speed projectile is in general a function of both Mach's and Reynolds' numbers but, because of the high values of the latter, compressibility is usually a considerably more important factor than viscosity. Thus it is to be expected that the forms of bodies of minimum drag will not necessarily be the same as those discussed in Chap. XII where viscosity alone was considered. The data shown in Fig. 239 illustrate the importance of using a sharp nose for a shell. More detailed information of this kind is presented graphically in Fig. 242, in which the drag coefficients of shells with different-shaped noses are plotted against Mach's number. The nose contours are defined by the radii shown in the accompanying table. It is apparent from these curves that an increase in the small radius at the nose leads to an appreciable increase in the value of C_D for Mach's number greater than unity, while, for velocities just below that of sound, small variations in this radius are of little importance. At much lower velocities, not shown in Fig. 242, the shape of the nose is of considerable importance in determining the drag of the shell. In this region viscosity is again the governing factor and the ideal shell has a well-rounded nose of the type represented by shell *a* of Fig. 242.

The shape of the base of the projectile is of considerable importance at subsonic velocities, and at these speeds the resistance is appreciably lowered by the addition of a streamlined tail. However, at supersonic velocities the shape of the tail does not have much effect on the drag. The reason for this lies in the fact that, when the acoustic velocity is reached in the wake, the pressure there drops nearly to zero so that this region is almost a perfect vacuum. A higher speed cannot produce any greater reduction in pressure in the wake and the eddy-making drag therefore remains

almost constant with further increases in speed and is practically independent of the shape of the tail.

The use of a fully streamlined tail on an artillery projectile presents certain mechanical difficulties in firing. Such a shape also has the aerodynamic disadvantage of being considerably more unstable than the flat-tailed shell. It is difficult to maintain its

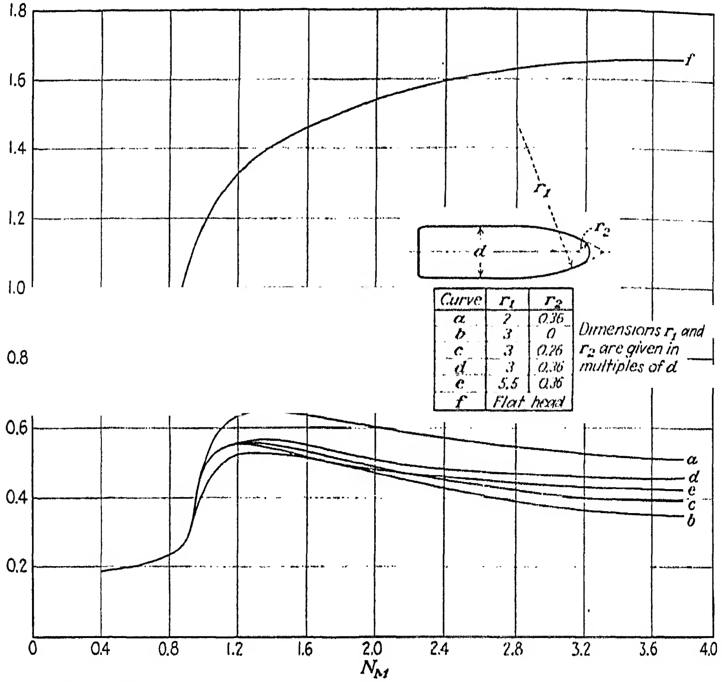


FIG. 242.—Effect of nose radius on resistance coefficient of projectiles. (*F. R. W. Hunt, "The Reaction of the Air to Artillery Projectiles," in "The Mechanical Properties of Fluids," Blackie & Son, Ltd., London,* 1923.)

path of flight in a vertical plane and the accuracy of gun fire with such shells is considerably reduced. A compromise solution to the problem has been obtained by the use of shells of a so-called boat-tail form, the rear ends having a slight taper of from 5 to 10 deg. for a short distance so that a longitudinal section resembles the plan view of a boat hull. These shells are very stable and show appreciably lower drag coefficients than the flat-tailed shapes.[1]

[1] For a more complete discussion of the application of fluid mechanics, the reader is referred to the article, "The Reaction of the Air to Artillery

188. Effects of Compressibility on the Lift and Drag of Airfoils.—Although the remarks of the preceding article were concerned with the effect of compressibility on the resistance of projectiles, the results given therein are in general applicable to bodies of all forms. In the case of lifting elements such as airfoils, the problem is further complicated by the fact that compressibility has an effect on the lift force as well as on the drag.

It may be shown that the presence of a lift force in general tends to aggravate the effect of compressibility. On the upper surface of an airfoil which is producing a certain positive lift as a result of its motion relative to the air, there is, as was shown in Art. 71, a considerable increase in local velocity over that of the undisturbed stream. For instance, it is not at all unusual to find experimentally that the negative pressure on the top of an airfoil may be as low as the pressure of the undisturbed stream diminished by four to five times the dynamic pressure $\rho_1 V_1^2/2$. Thus the pressure p_2 on the airfoil in terms of conditions at point 1 in the undisturbed stream is

$$p_2 = p_1 - n\left(\frac{\rho_1 V_1^2}{2}\right) \tag{38}$$

The substitution of this value in Eq. (18), page 368, gives for the square of the velocity at the point on the airfoil

$$V_2^2 = V_1^2 - \frac{2c_1^2}{k-1}\left\{\left[1 - \frac{n}{p_1}\left(\frac{\rho_1 V_1^2}{2}\right)\right]^{\frac{k-1}{k}} - 1\right\} \tag{39}$$

The value of local velocity at point 2 is readily found by taking the square root of both sides of Eq. (39). Putting $kp_1/\rho_1 = c_1^2$ and $V_1/c_1 = N_{M_1}$ in the term inside the brackets, this expression becomes

$$V_2 = \left\{V_1^2 - \frac{2c_1^2}{k-1}\left[\left(1 - \frac{nk}{2}(N_{M_1})^2\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} \tag{40}$$

Inasmuch as Mach's number appears to be a significant factor in connection with high-speed motion, it is now proposed to determine the value of this quantity for the conditions prevailing at the point on the airfoil. In the undisturbed stream the

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Projectiles" by F. R. W. Hunt, Chap. X, "The Mechanical Properties of Fluids," Blackie & Son, Ltd., London, 1923.

acoustic velocity is $c_1 = \sqrt{kp_1/\rho_1}$ and in this region Mach's number is

$$N_{M_1} = \frac{V_1}{c_1} \tag{41}$$

At point 2 both the pressure and density have values different from those existing at point 1. The acoustic velocity at point 2 is $c_2 = \sqrt{kp_2/\rho_2}$ and, after substituting the value of p_2 as given by Eq. (38) and making use of the adiabatic law, $p_1/\rho_1{}^k = p_2/\rho_2{}^k$, the value of this velocity is

$$c_2 = c_1\left[1 - \frac{nk}{2}(N_{M_1})^2\right]^{\frac{k-1}{2k}} \tag{42}$$

Mach's number at point 2 is the quotient of Eq. (40) divided by Eq. (42) and is equal to

$$N_{M_2} = \frac{V_2}{c_2} = \frac{V_1{}^2 - \frac{2c_1{}^2}{k-1}\left\{\left[1 - \frac{nk}{2}(N_{M_1})^2\right]^{\frac{k-1}{k}} - 1\right\}}{c_1\left[1 - \frac{nk}{2}(N_{M_1})^2\right]^{\frac{k-1}{2k}}}$$

or

$$N_{M_2} = \frac{\left[(N_{M_1})^2 - \frac{2}{k-1}\left\{\left[1 - \frac{nk}{2}(N_{M_1})^2\right]^{\frac{k-1}{k}} - 1\right\}\right]^{\frac{1}{2}}}{\left[1 - \frac{nk}{2}(N_{M_1})^2\right]^{\frac{k-1}{2k}}} \tag{43}$$

The two values of Mach's number given by Eqs. (41) and (43) are referred to, respectively, as the Mach's number of the undisturbed stream and the local value of Mach's number.

The significance of the result given in Eq. (43) is best brought out by means of a numerical example. The flow of air will be considered in which $c_1 = 1120$ ft. per sec., $p_1 = 2116.8$ lb. per sq. ft., $\rho_1 = 0.002378$ slugs per cu. ft., $k = 1.406$ and $n = 4$. The results of computations with Eq. (43) for several values of N_{M_1} are shown graphically in Fig. 243, in which N_{M_2} has been plotted against N_{M_1}. It appears from these calculations that the velocity at the low-pressure point on the airfoil exceeds the corresponding velocity of sound when the value of Mach's number for

the undisturbed stream is only 0.383 and the corresponding velocity is 429 ft. per sec.

The curve of Fig. 243 shows that, at a certain value of N_{M_1}, N_{M_2} tends to become infinitely large. This limiting value of N_{M_1}

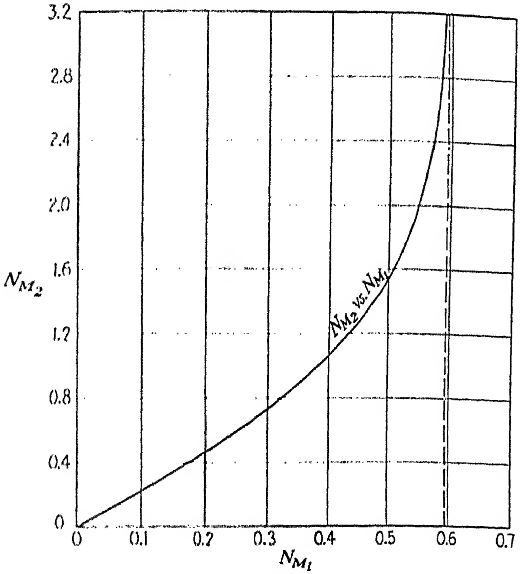


FIG. 243. Variation of local value of Mach's number with its value for an undisturbed airstream.

may be found from Eq. (43) by putting the denominator equal to zero. Its value is

$$N_{M_1} = \sqrt{\frac{2}{nk}}$$

and, for $n = 4$ and $k = 1.406$, $N_{M_1} = 0.594$. The corresponding value of velocity of the stream is

$$V_1 = c_1\sqrt{\frac{2}{nk}}$$

The local pressure at point 2 on the body may now be determined from Eq. (38) and is equal to

$$p_2 = p_1 - \frac{n\rho_1}{2}\frac{2p_1}{n\rho_1} = 0$$

The most important conclusion to be drawn from the above analysis is that, in the case of the flow of a gas past bodies such as airfoils, local increases in velocity caused by the body may produce reductions in pressure of such a nature that the local values of Mach's number approach the critical unit value much more rapidly than does the Mach's number for the undisturbed stream. Consequently the effects of compressibility become noticeable for values of N_{M_1} considerably less than unity.

Local effects of this type are found on symmetrical bodies such as artillery projectiles for which the air reaction is entirely a drag force, but they are by no means so severe as in the case of airfoils because the reductions in pressure are considerably smaller in magnitude. It seems logical to expect that at high speeds the drag of an airfoil will vary with velocity in much the same way as that of a projectile, with the exception that the effects of compressibility will be noticeable at lower values of the Mach's number for the undisturbed flow.

As to the effect of compressibility on lift, it may be concluded on the basis of the preceding discussion that this is in general a detrimental one. If the pressure at a point on the upper surface of an airfoil has dropped to zero, it is obvious that there can be no further increase in the suction produced by the element of surface at this point with an increase in speed. The lift therefore increases at a rate somewhat less than the square of the velocity and the lift coefficients computed on the basis of the usual "V-squared" law show a decrease in value.

Some studies of the flow of compressible fluids past solid bodies have been made by means of the methods of theoretical hydrodynamics but this work is for the most part extremely limited in its application. The exact effects of compressibility are best determined by experimental means, although a basic theory is of considerable value in correlating test data and in bringing out the significance of the most important factors.

189. The Compressibility Burble.—The remarks in the preceding paragraphs about the effects of compressibility on the lift and drag of airfoils have been verified by tests made in high-speed wind tunnels. Most of the work done in the United States has been carried out by the Bureau of Standards and the National Advisory Committee for Aeronautics and some typical results of tests made at the laboratories of the latter organization are shown

in Fig. 244. The sections on which the tests were made are typical of the kind used for the elements of propeller blades. It may be noted from the curves of Fig. 244 that, as the Mach's number of the airstream is increased, both the lift and drag coefficients for any angle of attack increase slowly until a value of N_{M_1} of about 0.8 is reached. Beyond this point the lift coefficient drops off very rapidly while the drag coefficients show a marked increase. This sudden change in the values of the lift and drag coefficients is more clearly shown by means of the curves

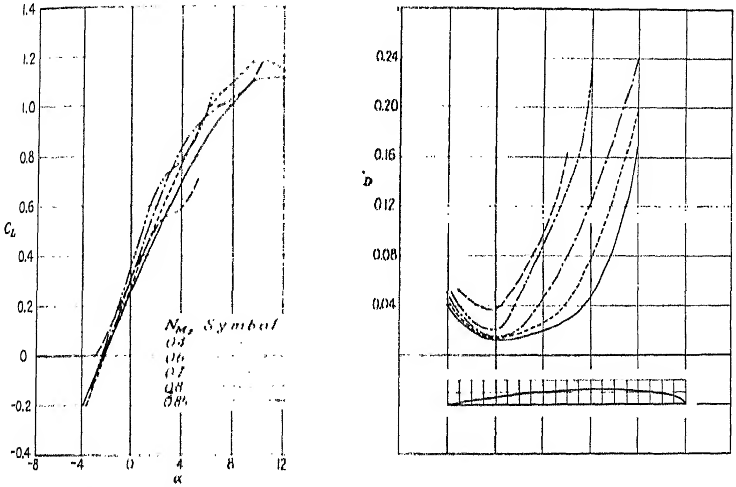


FIG. 244. Effects of compressibility on lift and drag coefficients of the 3R6 airfoil. (*J. Stack, The NACA High Speed Wind Tunnel and Tests of Six Propeller Sections, NACA Tech. Rept.* 463.)

of Fig. 245. In this figure the slopes of the lift curves of Fig. 244 are plotted as functions of Mach's number, that is, $dC_L/d\alpha$ with α in radians is plotted against N_{M_1}. The minimum drag coefficients $C_{D_{\min.}}$ are also plotted against N_{M_1}. In these tests the model extended completely across the air jet so that the coefficients shown correspond to wings of infinite aspect ratio and do not include the effects of tip vortices.

Schlieren photographs of the high-speed flow of air past airfoils show that, under the conditions for which this marked change in lift and drag coefficients begins, a shock wave originates at the nose of the airfoil. This phenomenon is known as the com-

pressibility burble[1] and is quite similar in character to the shock-wave formation found in the motion of high-speed projectiles. The compressibility burble is apparently caused by the fact that at some point on the airfoil surface the velocity has become sufficiently high to cause the pressure to drop to zero. The shock wave encloses a region at the rear of the airfoil in which the gas is

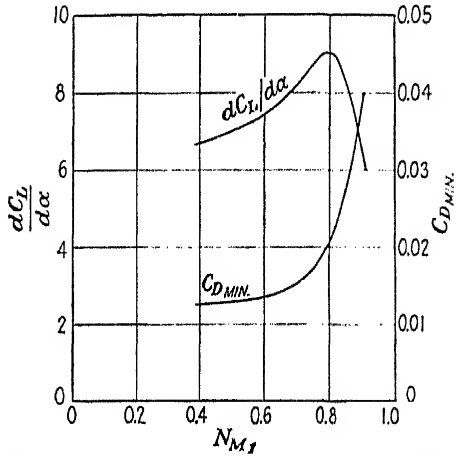


FIG. 245.—Variation of lift-curve slope and minimum drag coefficient with Mach's number for the 3R6 airfoil.

in an extremely rarefied state. If the speed of the stream is increased, this low-pressure region expands but at a rate which is lower than the rate of increase of the square of the velocity, thus producing the decrease in lift coefficient and increase in drag coefficient shown in Fig. 245.

There is some similarity between the conditions producing the compressibility burble in a gas and those leading to cavitation in the flow of liquids. Both occur at relatively high velocities, but the detailed natures of the flows involved in the two cases are quite dissimilar.

[1] STACK, J., The Compressibility Burble, *NACA Tech. Note* 543.

CHAPTER XIV

THERMODYNAMICS OF COMPRESSIBLE VISCOUS FLUIDS

190. Gases Considered as Compressible Viscous Fluids.—The discussions of Chap. XIII have all been based on the assumption that the fluids under consideration could be regarded as nonviscous. Although gases have relatively low coefficients of viscosity, the fact that extremely high velocities are often encountered makes it possible that frictional losses may appreciably modify the results previously obtained. The introduction of viscosity greatly complicates the theory and experimental methods are usually resorted to in order to determine the behavior of compressible viscous fluids. There are, however, a number of cases of the flow of gases in which both compressibility and viscosity can be taken into consideration without unduly complicating the analysis. This is particularly true of the flow in pipe lines of constant cross section. The theory to be presented in the next three articles is intended to serve as a foundation for a discussion of that subject.

191. Bernoulli's Theorem for the Flow of Gases.—When a stream of gas moves in contact with solid boundaries, the effect of viscosity is to produce a resistance to the motion which is responsible in part for the dissipation of the total energy content of the particles of fluid. If there are no losses due to friction or any other causes, then Bernoulli's theorem, as given by Eq. (13), page 367, is

$$\frac{dp}{\rho} + g\,dz + \frac{d(V^2)}{2} = 0$$

In this expression each term represents energy per unit mass. The energy relation for a unit weight of fluid may be obtained by dividing the above equation by g. After putting $\rho g = 1/v$, where v is the specific volume, the above equation is

$$v\,dp + dz + \frac{d(V^2)}{2g} = 0$$

When corrected for the loss in energy due to the resistance to flow, this equation becomes

$$v\,dp + dz + \frac{d(V^2)}{2g} + dE_f = 0 \tag{1}$$

in which dE_f is the energy lost by a unit weight of fluid. It is obvious that some knowledge of the way in which E_f varies along a streamline must be available before this equation can be integrated. In the case of a flow in a pipe the loss may be obtained from the resistance formulas developed in Chap. IX. However, before considering any special applications of this modified form of Bernoulli's theorem, the energy relations involved in the flow of gases will be discussed from another point of view.

192. The Thermodynamic Equations for Gas Flow.—In view of the fact that in engineering work the motion of gases is often closely connected with some process involving the interchange of heat, it seems advisable to consider the energy relations from the standpoint of thermodynamics. When energy in the form of heat is added to a gas, two changes may occur. First, the addition of heat may produce a rise of temperature and an increase in the sensible heat content or intrinsic energy of the gas; second, the gas may expand against the pressure on the walls of its container. This second form of the change of the original heat energy is known as external work and is present only when the walls of the gas container are flexible or movable. The calculation of the intrinsic energy will be discussed in the next article.

The magnitude of the external work done by a gas during any heat transfer may be readily calculated in terms of its pressure and specific volume by considering an infinitesimal element of the fluid in the form of a small cylinder of length l, of cross-sectional area dA and of unit weight. Suppose now that, as a result of the addition of heat, the element expands in the direction of its length to a new value $l + dl$ while the cross-sectional area remains unchanged. Then, if the pressure acting on the element is p, the work done by this expansion is $p\,dA\,dl = p\,dv$, since $dA\,dl = dv$, the change in volume of the element. Furthermore, since the element was originally assumed to be of unit weight, v represents the specific volume of the gas.

It is often convenient to represent the changes in pressure and specific volume of a gas that occur while it is doing external work

by means of a curve plotted on a system of rectangular coordinate axes, the abscissas representing the specific volume while the corresponding pressures are plotted as ordinates. Such an arrangement is illustrated in Fig. 246, the particular case shown therein being a typical cycle of operations on a gas. The cycle begins at point A where the gas has a certain initial pressure and specific volume. The pressure is then increased to that represented by point B while the volume remains constant, this process requiring the addition of heat. After reaching B the gas is allowed to expand first at constant pressure to C, after which the pressure decreases to the condition represented by D. The cycle is completed by compressing the gas at constant pressure from D back to the initial point A. As shown above, the external work done by a small element or particle of gas is $p\,dv$ so that in the cycle of operations shown in Fig. 246 the total work done by a unit weight of the gas is $\int p\,dv$ or simply the area enclosed by the curve $ABCDA$.

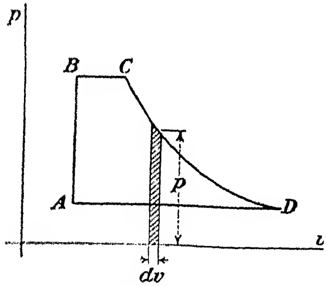


FIG. 246. Pressure-volume diagram for a gas cycle.

The relationship between heat added, intrinsic energy and external work may now be put in the form of an equation

$$dQ = dI + \frac{p\,dv}{J} \tag{2}$$

when applied to an infinitesimal particle of fluid of unit weight. In this expression dQ is the heat added externally, dI is the increase in intrinsic energy and $p\,dv/J$ is the external work done, J being the factor that converts energy in the form of work, measured by the product of force times displacement, into units of heat energy. In the English system this latter unit is known as the British thermal unit (B.t.u.) and the ratio between 1 B.t.u. and 1 ft. lb. is $J = 778$.

In the case of a flow in which resistance to motion is involved, the energy expended in overcoming resistance is converted into heat, which is then added to the gas just as though it came from an external source. When this loss is taken into account, Eq. (2) must be modified to read

$$dQ + \frac{dE_f}{J} = dI + \frac{p\,dv}{J} \tag{3}$$

In addition to this relation, another expression may be worked out which involves the kinetic and potential energies of the flow. In Fig. 247 is shown a portion of a stream tube and consideration is to be given to two sections, at the first of which the elevation is z_1, the pressure is p_1, the velocity is V_1, the specific volume is v_1 and the cross-sectional area of the tube is A_1. At the second section the corresponding values are z_2, p_2, V_2, v_2 and A_2. The gas is assumed to be flowing from point 1 to 2. If the weight of fluid flowing through the tube in unit time is W, then the equation of continuity may be written in the form

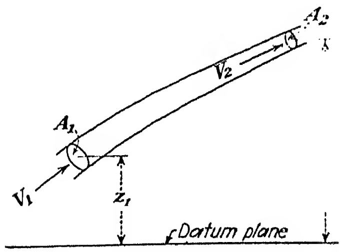


FIG. 247.—Stream tube for the flow of a gas.

$$W = \frac{A_1 V_1}{v_1} = \frac{A_2 V_2}{v_2} = \frac{AV}{v} \tag{4}$$

The difference in energy content of the fluid between the two levels considered may be determined in two different ways. This relation will first be determined by considering the differences in kinetic, intrinsic and potential energy between points 1 and 2 as well as the work done against resistance to flow. The kinetic energy at any point in the stream tube is $V^2/2g$ per unit of weight and, since this quantity is the velocity head, it will be represented by h. The difference in the kinetic energy of the fluid at the points 1 and 2 is then

$$h_1 \quad \frac{V_2^2}{2g} \quad \frac{V_1^2}{2g}$$

There is also a change in the intrinsic energy of the gas which may be written as

$$J(I_2 - I_1)$$

I_1 and I_2 being expressed in units of heat energy. The loss in energy due to resistance is E_f while the change in potential energy is $z_2 - z_1$. In all cases the energy is computed for a unit weight of the gas. A comparison of the total amounts of energy at the two sections of the stream tube shows that the energy expended

in moving a unit weight of the fluid from section 1 to section 2 is

$$E = h_2 - h_1 + J(I_2 - I_1) + z_2 - z_1 + E_f \tag{5}$$

The value of E may also be determined by considering the external forces that act on the fluid during the motion. In the first place Q units of heat may have been added from an outside source which, in units of work, is QJ. The work against resistance is converted into heat which is added to the gas so that the work equivalent of the total heat added is $QJ + E_f$ for each pound of gas. Now in a time dt the particles composing the lower face of the stream tube of Fig. 247 travel a distance $V_1\,dt$, and the pressure force acting on it, p_1A_1, does work equal to $p_1A_1V_1\,dt$. In a similar way the work done by the pressure force p_2A_2 acting in the opposite direction on area A_2 is $p_2A_2V_2\,dt$, so that the net work done on the gas between sections 1 and 2 in the time dt is

$$p_1A_1V_1\,dt - p_2A_2V_2\,dt$$

But from Eq. (4) $A_1V_1 = v_1W$ and $A_2V_2 = v_2W$ so that this net work is

$$(p_1v_1 - p_2v_2)W\,dt$$

During time dt the weight $W\,dt$ has entered the tube through section 1 and an equal amount has left the tube through section 2. The last expression may therefore be considered as the amount of work done on a weight of gas equal to $W\,dt$, so that for a unit weight the work done is

$$p_1v_1 - p_2v_2$$

The total amount of energy expended on a unit weight of fluid is thus equal to

$$E = p_1v_1 - p_2v_2 + QJ + E_f \tag{6}$$

When this value of E is equated to that given by Eq. (5), the term representing the work done against resistance disappears and the equation may be solved for Q with the result

$$Q = \frac{1}{J}\left[h_2 - h_1 + J(I_2 - I_1) + z_2 - z_1 + p_2v_2 - p_1v_1\right] \tag{7}$$

If the portion of the stream tube between sections 1 and 2 is of infinitesimal length, then the heat added externally is dQ

and the various differences on the right side of Eq. (7) may be replaced by their corresponding differentials so that

$$dQ = \frac{1}{J}\left[dh + J\,dI + dz + d(pv)\right] \tag{8}$$

If the values of dQ obtained from Eqs. (3) and (8) are equated, the expression

$$dh + dz + v\,dp + dE_f = 0 \tag{9}$$

is obtained, the differential $d(pv)$ having been put equal to $p\,dv + v\,dp$. Now recalling that $h = V^2/2g$, the velocity head, Eq. (9) may be written as

$$\frac{d(V^2)}{2g} + dz + v\,dp + dE_f = 0 \tag{10}$$

which is exactly the same as Bernoulli's theorem given by Eq. (1). It thus appears that the thermodynamic method of attacking the flow of gases and the mechanical method employed in the problems previously discussed are essentially the same because they lead to the same fundamental equation. This might have been anticipated since the equations obtained in each case are simply statements of the principle of conservation of energy. In studying the flow of gases there are certain types of problems, particularly those in which the loss is considered, in which the use of the thermodynamic method has some advantage. In such problems Bernoulli's theorem in the form of Eq. (10) will be employed along with the thermodynamic relationship given by Eq. (3).

Problem 371. Determine the external work done by a unit weight of gas in expanding from a pressure p_1 to a pressure p_2 if the expansion takes place isothermally. Express the results in terms of the initial and final pressures and the initial specific volume v_1.

372. Determine the external work done by a unit weight of gas in expanding adiabatically from a pressure p_1 to a pressure p_2 if the corresponding change in specific volume is from v_1 to v_2.

373. A cylinder contains 3.5 lb. of air at a pressure of 100 lb. per sq. in. abs. and at a temperature of 95°F. The air is allowed to expand isothermally until the volume is twice its original value. Determine the final pressure and the external work done by the gas.

374. Solve Prob. 373 when the expansion is adiabatic.

375. A compressor does 450,000 ft.-lb. of work on 1 lb. of a gas in compressing it adiabatically from a pressure of 15 lb. per sq. in. abs. The

density is initially 0.0014 slugs per cu. ft. and $k = 1.32$. Find the pressure and density after compression.

193. The Intrinsic Energy of Gases.—In the first part of Art. 192 it was shown that the external work done by a gas is equal to $p\,dv$ for a unit of weight. It is now necessary to find a similar expression for that portion of the heat added to a gas which is taken up in increasing its intrinsic energy. For the moment no distinction will be made between heat added to the gas from an external source and that generated in overcoming resistance, so Eq. (2) may be used. This expression, when applied to a finite volume of gas, becomes, after integration,

$$Q = I_2 - I_1 + \frac{1}{J}\int_{v_1}^{v_2} p\,dv \tag{11}$$

where the subscripts 1 and 2 denote the initial and final states, respectively. If heat is added to the gas and the volume is maintained constant, then obviously the external work is equal to zero and all the heat added will be utilized in raising the temperature. Under such conditions Eq. (11) becomes

$$Q = c_v(T_2 - T_1) = I_2 - I_1$$

in which c_v is the specific heat of the gas at constant volume and T_1 and T_2 are the initial and final absolute temperatures. In this case the change in intrinsic energy of the gas is measured directly by the change in temperature. This is true whether the heat is added at constant volume or otherwise. That is,

$$I_2 - I_1 = c_v(T_2 - T_1) \tag{12a}$$

or, in differential form,

$$dI = c_v\,dT \tag{12b}$$

The change in intrinsic energy is always directly proportional to the change in temperature.

When heat is added and the pressure is kept constant, Eq. (11), after integrating and substituting $I_2 - I_1$ from Eq. (12a), becomes

$$c_p(T_2 - T_1) = c_v(T_2 - T_1) + \frac{p(v_2 - v_1)}{J} \tag{13}$$

c_p being the specific heat at constant pressure. But the equation of condition for a perfect gas [Eq. (12), page 6] is $pv = RT$, so that

$$p(v_2 - v_1) = R(T_2 - T_1) \tag{14}$$

After substituting $p(v_2 - v_1)$ from Eq. (14) and dividing out the term $T_2 - T_1$, Eq. (13) becomes

$$c_p - c_v = \frac{R}{J} \tag{15}$$

Now the change in intrinsic energy in any type of expansion, found from Eq. (12a) by introducing the values of T_1 and T_2 from the equation of condition, is

$$I_2 - I_1 = \frac{c_v}{R}(p_2v_2 - p_1v_1) = \frac{c_v}{J}\frac{J}{R}(p_2v_2 - p_1v_1)$$

After substituting the value of J/R from Eq. (15), this expression becomes

$$I_2 - I_1 = \frac{c_v}{J(c_p - c_v)}(p_2v_2 - p_1v_1) = \frac{1}{J(k-1)}(p_2v_2 - p_1v_1) \tag{16a}$$

where $k = c_p/c_v$ is the ratio of specific heats at constant pressure and constant volume. This equation may also be written in the differential form

$$dI = \frac{d(pv)}{J(k-1)} \tag{16b}$$

By means of Eq. (16a) or (16b) the intrinsic energy change involved may be expressed in terms of the changes in pressure and specific volume. With these preliminary developments completed, special problems in the flow of gases may now be discussed.

Problem 376. What is the change in intrinsic energy of a gas if its temperature is increased from 60 to 220°F.? R is 55.0 ft. per °F. and $k = 1.40$.

377. Air at a pressure of 14.7 lb. per sq. in. abs. and a temperature of 60°F. absorbs 25 B.t.u. of heat per pound, during which process the temperature rises to 130°F. Determine the change in intrinsic energy and the external work per pound of air.

378. What is the change in intrinsic energy of 1 lb. of air if it expands adiabatically from a pressure of 210 lb. per sq. in. abs. and a temperature of 110°F. to a pressure of 15 lb. per sq. in. abs.? What is the amount of external work done by the gas during this expansion?

194. Low-velocity Flow in Pipes at Constant Temperature.—The motion of gases in tubes of variable cross-sectional area is too complex to be considered here, but the flow through pipes of constant cross section presents a problem that is relatively simple in its treatment. Let it be assumed that a pipe line is so located that the temperature of the gas remains essentially constant and that the velocities of flow are so small that the kinetic energy may be neglected. Bernoulli's theorem as represented by Eq. (10) then becomes

$$dz + v\,dp + dE_f = 0 \tag{17}$$

Now the loss due to resistance for a length of pipe dl, corresponding to the change in elevation dz, as given by Eq. (20), page 202, is

$$dE_f = f\,\frac{dl}{D}\,\frac{V^2}{2g}$$

where f in general is a function of the Reynolds' number. The symbol D is used here for the diameter in order to avoid confusion with the differential symbol d. As long as the temperature is constant it may be shown that the Reynolds' number and therefore the value of f remain unchanged. At constant temperature the coefficient of viscosity is constant and the Reynolds' number will be affected only by changes in the velocity and density. The relation between these last two quantities may be found from the equation of continuity

$$W = \frac{AV}{v} \qquad \text{or} \qquad v = \frac{AV}{W}$$

The density is $\rho = 1/gv$ and substituting the above value of v gives $\rho = W/AVg$. The Reynolds' number is then equal to

$$N_R = \frac{\rho VD}{\mu} = \frac{WD}{Ag\mu}$$

This equation shows that, for a given weight discharge, pipe diameter and temperature, N_R is constant. Hence the friction factor f is also constant.

The heat relationship for this problem, as given by Eq. (3), is

$$dQ + \frac{dE_f}{J} = \frac{p\,dv}{J} \tag{18}$$

dI being equal to zero when the temperature is constant. The right-hand member of this expression may be written as

$$\frac{p\,dv}{J} = \frac{(p\,dv + v\,dp - v\,dp)}{J} = \frac{d(pv)}{J} - \frac{v\,dp}{J}$$

But from Eq. (16b), $\dfrac{d(pv)}{J} = (k - 1)dI = 0$ since in this case $dI = 0$. Then $p\,dv/J = -v\,dp/J$. Substituting this in Eq. (18), it becomes

$$dQ + \frac{dE_f}{J} = -\frac{v\,dp}{J} \tag{19}$$

A comparison of Eqs. (17) and (19) shows that dQ is equal to dz/J, that is, the energy added in the form of heat from an external source must be exactly equal to the change in potential energy due to the rise in the pipe line. If the pipe line were horizontal, then $dz = 0$ and no heat would be added. In these cases the heat generated internally in overcoming the resistance to flow is completely utilized in doing external work on the gas and produces changes in pressure and density in the pipe line.

In order to determine the value of the velocity or pressure at any point in the pipe, it is necessary to introduce the equation of condition of the gas for the isothermal state, that is,

$$pv = p_1v_1 = p_2v_2 = RT \tag{20}$$

It will be assumed that the pipe line is straight and slopes upward at an angle α with the horizontal so that the relation between elevation and length is

$$z = l \sin \alpha \qquad \text{or} \qquad dz = dl \sin \alpha \tag{21}$$

It is now desired to evaluate the term $v\,dp$ in Eq. (17). From the equation of continuity the specific volume is $v = AV/W$. The pressure at any point, as given by Eq. (20), is then

$$p = \frac{p_1v_1}{v} = p_1v_1\frac{W}{AV}$$

the subscript 1 being used to indicate conditions at the entrance to the pipe where l and z are taken as equal to zero. The differential of the pressure obtained from this last expression is

$$dp = -p_1v_1\frac{W\,dV}{AV^2}$$

and finally the desired term is

$$v\,dp = \left(\frac{AV}{W}\right)\left(\frac{-p_1v_1W\,dV}{AV^2}\right) = -p_1v_1\frac{dV}{V}$$

The introduction of this expression into Eq. (17), along with dz from Eq. (21) and the value of dE_f, gives a differential equation in terms of the variables l and V in the form

$$dl \sin\alpha - p_1v_1\frac{dV}{V} + f\frac{dl}{D}\frac{V^2}{2g} = 0 \qquad (22)$$

This equation is readily put into the form

$$dl = \left[\frac{p_1v_1}{V\left(\dfrac{f}{D}\dfrac{V^2}{2g} + \sin\alpha\right)}\right]dV$$

which can be integrated, and for the limits 0 and l, where the velocities are V_1 and V_2, respectively, it becomes

$$l = \frac{p_1v_1}{2\sin\alpha}\log_e\frac{V_2^2}{V_1^2}\left(\frac{\dfrac{f}{D}\dfrac{V_1^2}{2g} + \sin\alpha}{\dfrac{f}{D}\dfrac{V_2^2}{2g} + \sin\alpha}\right) \qquad (23)$$

This result may also be given in terms of the initial and final pressures by means of a simple transformation. From the equation of continuity

$$W = \frac{A_1V_1}{v_1} = \frac{A_2V_2}{v_2}$$

and since $A_1 = A_2$ the velocity at the outlet is $V_2 = V_1v_2/v_1$. But from Eq. (20) $v_2/v_1 = p_1/p_2$ and therefore $V_2 = p_1V_1/p_2$. Also from the equation of continuity $V_1 = Wv_1/A$ so that $V_2 = Wv_1p_1/Ap_2$. The substitution of these values in Eq. (23) gives the following equation for the length of pipe between the points where the pressures are p_1 and p_2:

$$l = \frac{p_1v_1}{2\sin\alpha}\log_e\left(\frac{p_1^2\sin\alpha + \dfrac{fW^2v_1^2p_1^2}{2gDA^2}}{p_2^2\sin\alpha + \dfrac{fW^2v_1^2p_1^2}{2gDA^2}}\right) \qquad (24)$$

For the special case of a pipe with its axis vertical and the gas flowing upward, $\alpha = 90$ deg. and Eqs. (23) and (24) become

$$l = \frac{p_1 v_1}{2} \log_e \left(\frac{V_2^2}{V_1^2} \frac{\frac{f}{D}\frac{V_1^2}{2g} + 1}{\frac{f}{D}\frac{V_2^2}{2g} + 1} \right) \tag{25}$$

and

$$l = \frac{p_1 v_1}{2} \log_e \frac{\left(p_1^2 + \frac{fW^2 v_1^2 p_1^2}{2gDA^2}\right)}{p_2^2 + \frac{fW^2 v_1^2 p_1^2}{2gDA^2}} \tag{26}$$

These formulas might be considered as applicable to the calculations of gas or air flow in chimneys and mine shafts.

The case of a horizontal pipe ($\alpha = 0$) leads to an indeterminate expression of the form 0/0 for Eqs. (23) and (24); while the limiting values may be found, it is simpler to return to Eq. (22) which now becomes

$$-p_1 v_1 \frac{dV}{V} + f \frac{dl}{D} \frac{V^2}{2g} = 0 \tag{27}$$

Dividing through by $p_1 v_1 V^2$, this expression becomes

$$\frac{dV}{V^3} \qquad \frac{f\,dl}{2gDp_1v_1}$$

and, after integration and substitution of the proper limits and noting that $p_1 v_1$ is constant,

$$\frac{1}{V_2^2} - \frac{1}{V_1^2} + \frac{fl}{gDp_1v_1} = 0$$

Making use of the relations between pressures and velocities, as was done in transforming Eq. (23) to Eq. (24), the result obtained is

$$\frac{p_1^2 - p_2^2}{p_1^2} = \frac{flV_1^2}{gDp_1v_1} \tag{28}$$

The pressure at point 2 is

$$p_2 = p_1\left(1 - \frac{flV_1W}{gDAp_1}\right)^{1/2} \tag{29}$$

These special cases of the horizontal and vertical pipes have been worked out by Kemler[1] but his solutions are presented in a slightly different form. When the velocities of flow are such that the kinetic energy cannot be neglected, the differential equation to be solved becomes somewhat more complicated and the solution of this problem will not be discussed here.[2]

Problem 379. Air flows through a smooth 4-in. horizontal pipe 1000 ft. long, the inlet conditions being as follows: pressure $p_1 = 125$ lb. per sq. in. abs., temperature $t_1 = 80°$F., velocity $V_1 = 120$ ft. per sec. Determine the pressure drop for every 250 ft. of pipe length when (*a*) the fluid is assumed incompressible, (*b*) the fluid is compressible and the temperature is constant.

380. Show by expansion in a series that for small values of $flV_1W/gDAp_1$ the pressure drop in a horizontal pipe line with isothermal flow is approximately the same as that given by the Darcy formula based on the assumption of an incompressible fluid.

381. A gas enters a smooth 3-in. horizontal pipe at 175 lb. per sq. in. gage, a velocity of 95 ft. per sec. and a density of 0.040 slugs per cu. ft. The pipe discharges into the open atmosphere. What is the maximum allowable length of the pipe? $R = 35.1$ ft. per °F. and $\mu = 4.53 \times 10^{-7}$ slugs per ft. sec.

382. Gas enters the base of a 3-ft. stack at 175°F., the rate of delivery being 9 lb. per sec. The pressure at the upper end is atmospheric (14.70 lb. per sq. in. abs.) while at the entrance it is 14.72 lb. per sq. in. abs. Determine the maximum allowable length of stack and the inlet and exit velocities with this length. $R = 40.0$ ft. per °F. and $f = 0.0227$.

195. Flow of Gas in Insulated Pipes.—When gas flows through a perfectly insulated pipe no heat can be added or withdrawn from the gas through the pipe walls. Thus the thermodynamic process involved is an adiabatic one. The heat generated in overcoming the resistance to the flow is, however, absorbed by the gas and consequently the relation between pressure and specific volume is not of the form $pv^k =$ constant. The general case of an inclined pipe in which the kinetic energy of the flow is taken into account involves the solution of a rather complicated differential equation[3] and only the case of a horizontal pipe will be discussed here. In this problem the change in the intrinsic energy of the gas is not equal to zero but has the value

[1] KEMLER, E., A Study of the Data on the Flow of Fluids in Pipes, *Trans. A.S.M.E., Hydraulics Div.*, 1933.

[2] STODOLA, A., "Steam and Gas Turbines," vol. I, p. 60; vol. II, p. 1025, McGraw-Hill Book Company, Inc., New York, 1927.

[3] GRASHOF, F., "Theoretische Maschinenlehre," vol. I, p. 594, Leopold Voss, Leipzig, 1875.

given by Eq. (16*b*). Noting that $dQ = 0$, the fundamental heat-energy relation of Eq. (3) becomes

$$dE_f = \frac{d(pv)}{(k-1)} + p\,dv$$
$$= \frac{(kp\,dv + v\,dp)}{k-1}$$

For a horizontal pipe Bernoulli's theorem from Eq. (10) is

$$\frac{d(V^2)}{2g} + v\,dp + dE_f = 0 \tag{30}$$

and, with the introduction into Eq. (30) of the above value of dE_f, this expression becomes

$$\frac{d(V^2)}{2g} + \frac{k\,d(pv)}{k-1} = 0$$

or

$$d(V^2) = -\frac{2kg}{k-1}\,d(pv)$$

The integration of this equation between the limits corresponding to the inlet of the pipe and to any other point gives the result

$$V^2 - V_1^2 = \frac{2kg}{k-1}(p_1v_1 - pv) \tag{31}$$

The frictional-loss term is now assumed to have the same form as in the constant-temperature problem of the preceding article; in other words, the friction factor f is still regarded as a constant. Under this assumption Eq. (30) becomes

$$\frac{d(V^2)}{2g} + v\,dp + f\frac{dL}{D}\frac{V^2}{2g} = 0$$

or, after adding and subtracting $p\,dv$,

$$\frac{d(V^2)}{2g} + d(pv) - p\,dv + f\frac{dL}{D}\frac{V^2}{2g} = 0 \tag{32}$$

The value of pv as obtained from Eq. (31) is

$$pv = p_1v_1 - \frac{(k-1)}{2kg}(V^2 - V_1^2) \tag{33}$$

It has already been mentioned that the pressure-volume relationship for this flow is not of the type pv^k = constant. The actual form of the equation of state may be obtained from Eq. (33) and is

$$pv + \frac{(k-1)V^2}{2kg} = p_1v_1 + \frac{(k-1)V_1^2}{2kg} \qquad (34)$$

or

$$pv\left[1 + \frac{(k-1)V^2}{2kgpv}\right] = \text{constant}$$

Now the quantity

$$kgpv = \frac{kp}{\rho} = c^2$$

c being the acoustic velocity corresponding to the pressure p. Thus the equation of condition becomes

$$pv\left(1 + \frac{k-1}{2}N_M^2\right) = \text{constant} \qquad (35)$$

in which N_M is Mach's number. If N_M is small, the relation between specific volume and pressure is approximately of the form

$$pv = \text{constant}$$

and, on comparison with the general gas law

$$pv = RT$$

it appears that this would be a case of constant temperature. In many cases the term involving Mach's number is not negligible and the pressure-volume relation is correctly given by either Eq. (34) or Eq. (35).

As an example, the pressure-volume diagram for the case of methane is shown in Fig. 248, the values of k and R being taken as 1.32 and 96.31, respectively. The initial conditions are assumed to be as follows: p_1 = 150 lb. per sq. in., t_1 = 100°F., $V_1/c_1 = N_{M_1} = 0.075$. The pipe diameter is 6 in. From the equation

$$p_1v_1 = RT_1$$

in which T_1 is the absolute initial temperature, the initial specific volume is found to be

$$v_1 = \frac{RT_1}{p_1} = 2.495 \text{ cu. ft./lb.}$$

For the necessary computations Eq. (34) may be used but it is first advantageous to introduce the weight discharge in place of the velocities. From the equation of continuity $V = Wv/A$, so that the initial velocity is $V_1 = Wv_1/A$. The substitution of these values in Eq. (34) gives

$$pv + \frac{(k-1)W^2v^2}{2kgA^2} = p_1v_1 + \frac{(k-1)W^2v_1^2}{2kgA^2} \tag{36}$$

In order to determine W it is necessary to find the value of V_1 from the given value of N_{M_1}. The acoustic velocity at the entrance to the pipe is

$$c_1 = \sqrt{kgp_1v_1} = 1512 \text{ ft./sec.}$$

so that

$$V_1 = N_{M_1}c_1 = 0.075 \times 1512 = 113.5 \text{ ft./sec.}$$

For a 6-in. pipe the cross-sectional area is 0.196 sq. ft. and the weight discharge is

$$W = \frac{AV_1}{v_1} = \frac{0.196 \times 113.5}{2.495} = 8.90 \text{ lb./sec.}$$

The curves for isothermal and adiabatic expansion represented by the equations pv = constant and pv^k = constant, respectively, are also shown in Fig. 248 for the same initial conditions as given above. These curves show that, except for relatively low pressures, the gas follows the isothermal curve very closely so that the assumption of a constant friction factor f is probably not far from the truth. The third curve representing the adiabatic expansion falls considerably below the other two and cannot be considered a good approximation of that given by the exact equation.

196. Limiting Conditions for Gas Flow in Insulated Pipes.—It will be noted on examination of the pressure-volume curves of Fig. 248 that the curve representing the exact condition of the gas as it flows through the pipe intersects the volume axis at a finite value of v when the pressure drops to zero. The numerical value of this abscissa may readily be found by putting p equal to zero in Eq. (36). For the example illustrated in Fig. 248 the

specific volume is 83.3 cu. ft. per lb. when the pressure has attained the absolute zero value. Such a condition, of course, cannot exist, for as the pressure approaches zero the specific volume must become infinitely large, as is the case for the isothermal and adiabatic expansions. The equation of condition

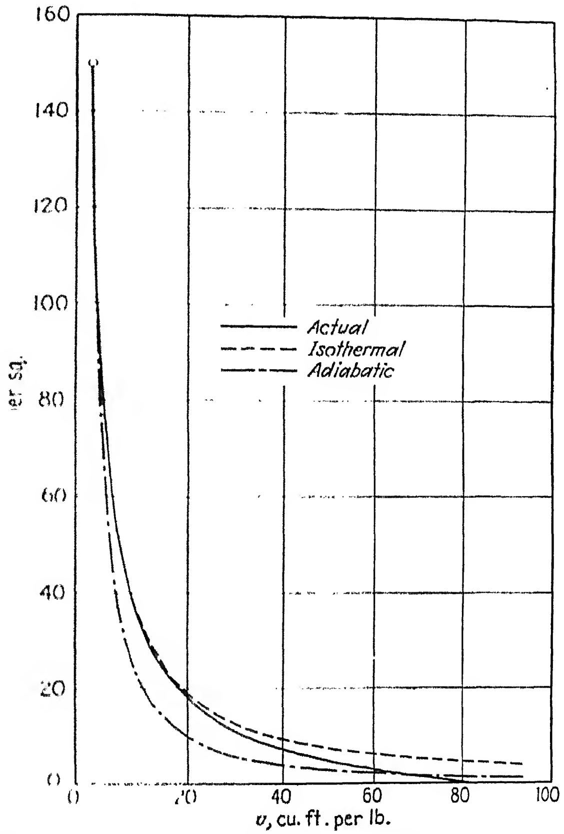


FIG. 248. Pressure-volume diagram for gas flow in an insulated pipe.

for gas flowing through a pipe, as developed in the preceding article, is therefore incorrect in the neighborhood of zero pressure.

This problem of gas flow in an insulated pipe has been classified as adiabatic because no heat is added or withdrawn externally from the system. However, the absorption by the gas of the heat generated by friction makes the pressure-volume relation of

Eq. (36) different in character from the usual adiabatic expansion represented by the equation

$$pv^k = B = \text{constant} \tag{37}$$

If there were no heat generated by friction, then Eq. (37) would correctly represent the relation between the pressure and specific volume at various points along the pipe.

The relation given by Eq. (36) is not valid at zero pressure and, as will be shown presently, cannot be valid beyond a certain limiting pressure. In order to determine this limiting value, the expansion represented by the "actual" curve shown in Fig. 248

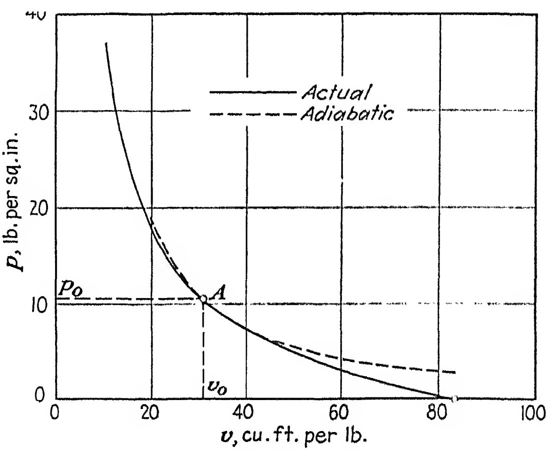


FIG. 249.—Determination of limiting conditions in an insulated pipe.

and by Eq. (36) may again be considered. For convenience the lower portion of this curve is drawn again in Fig. 249. Let it be supposed that point A on this curve at which the pressure is p_0 and the specific volume is v_0 represents a condition at which the friction loss has become zero. Since there can be no gain in energy due to friction, the expansion, if it is to continue to still lower pressures, must follow the limiting adiabatic law of Eq. (37), the constant B now being determined by conditions at A. The nature of the expansion in the pipe between the inlet and the point at which the pressure is p_0 is then determined by Eq. (36) while beyond this point it is given by Eq. (37). The curve represented by Eq. (37) is also shown in Fig. 249.

The changes in pressure and specific volume which occur in the gas as it flows through the pipe are normally continuous in character. Therefore the slopes of the two expansion curves must be the same at the point A, which they have in common. Furthermore, it is obvious that the friction loss in a pipe can never be completely reduced to zero except at the exit end of the pipe where the pipe wall ends. The pressure p_0 and the specific volume v_0 must therefore correspond to conditions at the discharge end of the pipe, represented by point A of Fig. 249.

The value of the limiting pressure and the corresponding specific volume may be found by equating the slopes of the two curves represented by Eqs. (36) and (37). In the equation of the actual flow, the constant coefficient $\dfrac{(k-1)W^2}{2kgA^2}$ is put equal to b for simplicity so that Eq. (36) becomes

$$pv = p_1v_1 + bv_1^2 - bv^2 \tag{38a}$$

from which

$$p = \frac{p_1v_1 + bv_1^2}{v} - bv \tag{38b}$$

The derivative of p with respect to v is then

$$\frac{dp}{dv} = -\frac{(p_1v_1 + bv_1^2)}{v^2} - b \tag{39}$$

For the limiting adiabatic expansion the pressure as obtained from Eq. (37) is

$$p = \frac{B}{v^k}$$

so that

$$\frac{dp}{dv} = -\frac{Bk}{v^{k+1}} = -\frac{kp}{v} \tag{40}$$

The limiting values of the pressure and specific volume, p_0 and v_0, respectively, are now obtained by equating the expressions for dp/dv as given by Eqs. (39) and (40) and at the same time putting $p = p_0$ and $v = v_0$. The result is

$$\frac{kp_0}{v_0} = \frac{p_1v_1 + bv_1^2}{v_0^2} + b$$

or

$$p_0 v_0 = \frac{p_1 v_1 + b v_1^2}{k} + \frac{bv}{k} \tag{41}$$

This value of $p_0 v_0$ may be equated to that obtained from Eq. (38*a*), from which an expression for the ratio of the final and initial values of the specific volume is obtained in the form

$$\left(\frac{v_0}{v_1}\right)^2 = \left(\frac{k-1}{k+1}\right)\left(1 + \frac{p_1}{b v_1}\right)$$

The original value of b is now introduced in this expression and at the same time the ratio W/A is replaced by its equivalent $\frac{V_1}{v_1}$ so that $b = \frac{(k-1)V_1^2}{2kgv_1^2}$ and the value of the specific volume ratio becomes

$$\frac{v_0}{v_1} = \left\{\left(\frac{k-1}{k+1}\right)\left[1 + \frac{2kgp_1v_1}{(k-1)V_1}\right]\right\}^{1/2}$$

The numerator of the second term in the brackets will be recognized at once as being equal to twice the square of the acoustic velocity at the entrance to the pipe; on writing c_1 for this acoustic velocity and letting the ratio $V_1/c_1 = N_{M_1}$, the initial value of Mach's number, the final result is

$$\frac{v_0}{v_1} = \left\{\left(\frac{k-1}{k+1}\right)\left[1 + \frac{\quad}{(k-1)N_{M_1}^2}\right.\right. \tag{42}$$

The value of the ratio of the final and initial pressures is found in a similar manner by solving Eq. (41) for p_0/p_1, that is,

$$\frac{p_0}{p_1} = \frac{1}{k}\left(\frac{v_1}{v_0} + \frac{b v_1^2}{p_1 v_0} + \frac{b v_0}{p_1}\right)$$

or

$$\frac{p_0}{p_1} = \frac{1}{k}\left[\left(1 + \frac{b v_1}{p_1}\right)\frac{v_1}{v_0} + \frac{b v_1}{p_1}\frac{v_0}{v_1}\right]$$

As in the preceding calculations the value of b is now substituted in this equation and the initial acoustic velocity $c_1 = \sqrt{kgp_1v_1}$ is also introduced, the result being

$$\frac{p_0}{p_1} = \frac{1}{k}\left\{\left[1 + \frac{(k-1)V_1^2}{2c_1^2}\right]\frac{v_1}{v_0} + \frac{(k-1)V_1^2}{2c_1^2}\frac{v_0}{v_1}\right\}$$

Putting $V_1/c_1 = N_{M_1}$, this may also be written in the form

$$\frac{p_0}{p_1} = \frac{v_0}{kv_1}\left\{1 - \frac{(k-1)N_{M_1}^2}{2}\left(\frac{v_1}{v_0}\right)^2 + \frac{(k-1)N_{M_1}^2}{2}\right\}$$

The last step in these calculations is to substitute the value of v_0/v_1 as given by Eq. (42). After clearing of fractions and simplifying, the above expression becomes

$$\frac{p_0}{p_1} = N_{M_1}^2\left\{\left(\frac{k-1}{k+1}\right)\left[1 + \frac{2}{(k-1)N_{M_1}^2}\right]\right\}^{\frac{1}{2}} \tag{43}$$

The ratio between the values of the velocity and specific volume of the gas at any point in the pipe is a constant, as may be shown by writing the equation of continuity in the form

$$\frac{W}{A} = \frac{V}{v} = \frac{V_1}{v_1} = \frac{V_0}{v_0}$$

both the weight and the area being constants. Thus

$$\frac{V_0}{V_1} = \frac{v_0}{v_1}$$

so that the ratio between the final and initial velocities is also given by Eq. (42). Now the acoustic velocity at the exit of the pipe is

$$c_0 = \sqrt{kgp_0v_0} = \left(kgp_1v_1\frac{p_0v_0}{p_1v_1}\right)^{\frac{1}{2}} = c_1\sqrt{\frac{p_0v_0}{p_1v_1}}$$

When the specific volume and pressure ratios, as given by Eqs. (42) and (43), are substituted in the above equation, the expression for the acoustic velocity at the exit becomes

$$V_1\left\{\left(\frac{k-1}{k+1}\right)\left[1 + \frac{2}{(k-1)N_{M_1}^2}\right]\right\}^{\frac{1}{2}}$$

But this is exactly the same as the limiting velocity obtained from Eq. (42). It thus appears that when a compressible gas flows through a pipe the pressure decreases until the velocity of the gas and the acoustic velocity at that pressure and corresponding specific volume are equal. A further decrease in the back pressure on the exit of the pipe would not modify the flow in any

way for the gas is moving at a higher velocity than the rate at which this decrease of pressure can be carried back into the pipe.

It will be noted that the expressions obtained for the ratios of the initial and final values of the specific volume, velocity and pressure are functions only of N_{M_1}, the initial value of Mach's number, and k, the ratio of specific heats. Values of these ratios for the case where $k = 1.32$, applicable to the study of the flow of such gases as carbon dioxide, ammonia and methane, and

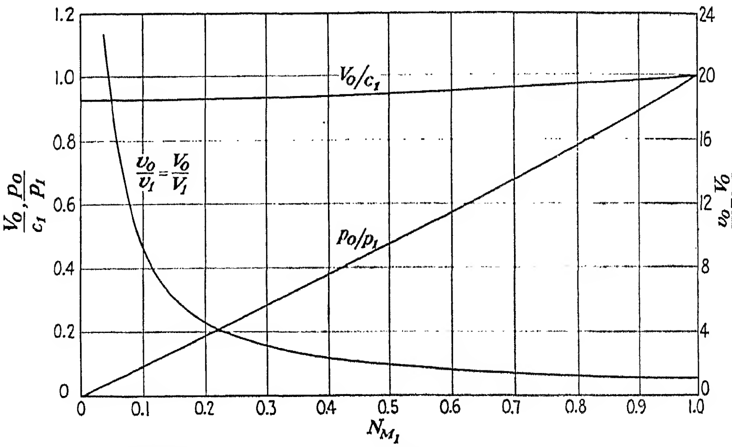


FIG. 250.—Limiting values of velocity, pressure and specific volume ratios for an insulated pipe as functions of Mach's number.

approximately to acetylene ($k = 1.28$), are shown by the curves in Fig. 250 for values of N_{M_1} running from zero to unity.

For the pipe flow previously discussed, the pressure-volume curve for which is shown in Fig. 248, $N_{M_1} = 0.075$. The limiting values of the specific volume, velocity and pressure ratios as determined from Fig. 250 are $v_0/v_1 = V_0/V_1 = 12.39$ and $p_0/p_1 = 0.0695$, and the conditions at the inlet and exit ends of the pipe are then as follows:

	Inlet end	Exit end
Pressure..........	150 lb. per sq. in.	10.42 lb. per sq. in.
Velocity..........	113.5 ft. per sec.	1404 ft. per sec.
Acoustic velocity..	1512 ft. per sec.	1404 ft. per sec.
Specific volume....	2.495 cu. ft. per lb.	30.90 cu. ft. per lb.

197. Conditions in the Interior of an Insulated Pipe.—The discussion of the previous article was entirely concerned with the determination of the conditions existing in a gas at the exit end of an insulated pipe through which the gas is flowing. It is now proposed to determine the length of pipe necessary to satisfy these conditions and also to investigate the variation in pressure, specific volume and velocity at points within the pipe. This may be accomplished by returning to a consideration of Eq. (33), which for convenience is rewritten here. It is

$$pv = p_1v_1 - \left(\frac{k-1}{2kg}\right)(V^2 - V_1^2) \tag{44}$$

From this expression the two terms $d(pv)$ and $p\,dv$ which appear in Eq. (32) may be determined. The first of these terms is simply the differential of Eq. (44), that is,

$$d(pv) = -\frac{(k-1)V\,dV}{kg} \tag{45}$$

The value of the second term is determined by calculating the specific volume from the equation of continuity, that is, $v = AV/W$. The differential of this expression is

$$dv = A\,dV/W = v\,dV/V,$$

so that

$$p\,dv = pv\frac{dV}{V}$$

or, using expression (44) for pv,

$$p\,dv = \left[p_1v_1 - \frac{(k-1)}{2kg}(V^2 - V_1^2)\right]\frac{dV}{V} \tag{46}$$

The substitution of expressions (45) and (46) in Eq. (32) gives

$$\frac{V\,dV}{g} - \frac{(k-1)V\,dV}{kg} - \left[\frac{2kgp_1v_1 - (k-1)(V^2 - V_1^2)}{2kg}\right]\frac{dV}{V} + \frac{f\,dl}{D}\frac{V^2}{2g} = 0$$

After dividing through by V^2/g and collecting terms, this equation becomes

$$\frac{f\,dl}{2D} = -\frac{(k+1)}{2k}\frac{dV}{V} + \frac{1}{k}\left[kgp_1v_1 + \frac{(k-1)}{2}V_1^2\right]\frac{dV}{V^3}$$

which may be integrated directly between the points 0 and l at which the velocities are respectively V_1 and V. The result is

$$\frac{fl}{D} = -\frac{(k+1)}{k}\log_e\frac{V}{V_1} + \frac{1}{k}\left[c_1^2 + \frac{(k-1)}{2}V_1^2\right]\left(\frac{1}{V_1^2} - \frac{1}{V^2}\right)$$

in which the combination kgp_1v_1 has been put equal to c_1^2. This may also be written in the form

$$\frac{fl}{D} = -\frac{(k+1)}{2k}\log_e\frac{V^2}{V_1^2} + \frac{1}{k}\left(\frac{1}{N_{M_1}^2} + \frac{k-1}{2}\right)\left(1 - \frac{V_1^2}{V^2}\right) \quad (47)$$

where $N_{M_1} = V_1/c_1$.

From Eq. (47) it is now possible to calculate the values of the quantity fl/D for a given value of N_{M_1} and for a series of values of V/V_1, this latter ratio being equal to v/v_1. If the friction factor f is known, it is a simple matter to determine the length of a given pipe corresponding to any particular velocity. The value of the length of pipe from the inlet to the exit for a given set of inlet conditions may be found by introducing the limiting value of V_0/V_1 from Eq. (42).

The ratio of the pressure at any point in the pipe to the initial value may be found by returning to the equation of state in the form of Eq. (38*b*). On dividing by the initial pressure p_1, this becomes

$$\frac{p}{p_1} = \frac{v_1}{v} + \frac{bv_1^2}{p_1v} - \frac{bv}{p_1}$$

The introduction of the value of b and the initial values of the acoustic velocity and Mach's number and the substitution of V/V_1 for v/v_1 make it possible to write this expression in the form

$$\frac{p}{p_1} = \frac{V_1}{V}\left[1 + \frac{(k-1)N_{M_1}^2}{2}\left(1 - \frac{V^2}{V_1^2}\right)\right] \quad (48)$$

The complete solution to the problem of the flow of gas through a pipe is now contained in Eqs. (42), (43), (47) and (48). A convenient procedure for using these equations is to make up a set of graphs from which the desired values can be readily determined for any given set of initial conditions. Figure 251 shows a series of curves representing $v/v_1 = V/V_1$ plotted as a function of fl/D for several values of N_{M_1}, these curves being

obtained by substituting values of V/V_1 in Eq. (47). The limiting value of V/V_1 is obtained from Eq. (42), being equal to v_0/v_1. The dotted curve in Fig. 251 represents these limiting values of the velocity ratio plotted against the corresponding values of fl/D. As in the case of the data shown in Fig. 250, the constant k has been taken as 1.32.

A similar set of curves showing the variation of the pressure ratio p/p_1 with fl/D for several values of N_{M_1} is plotted in

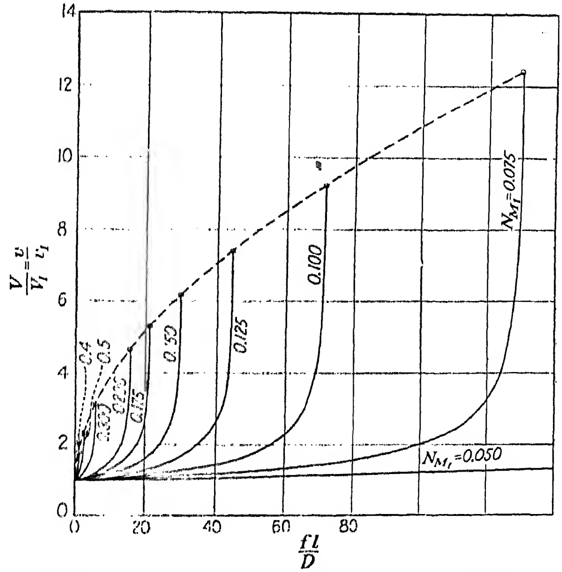


FIG. 251.—Variation of specific volume or velocity ratio with Mach's number and fl/D for an insulated pipe.

Fig. 252, the calculations being based on Eqs. (47) and (48). The limiting values of the pressure ratio as obtained from Eq. (43) are indicated by the dotted curve. It will be noted that in both Figs. 251 and 252 the slopes of the curves tend to become infinite as the limiting condition is approached.

The use of these curves is illustrated by the example that follows. Suppose that the gas in question is methane and that the initial conditions are those of the example shown in Fig. 248, that is, $p_1 = 150$ lb. per sq. in., $v_1 = 2.495$ cu. ft. per lb., $t_1 = 100$°F., $N_{M_1} = 0.075$, $c_1 = 1512$ ft. per sec., $V_1 = 113.5$ ft. per sec. and $W = 8.90$ lb. per sec. The pipe is made of smooth

steel and has a diameter of 6 in. The final conditions have already been determined from Fig. 250 and are given on page 412. The limiting values of the pressure, specific volume and velocity are, respectively, $p_0 = 10.42$ lb. per sq. in., $v_0 = 30.90$ cu. ft per

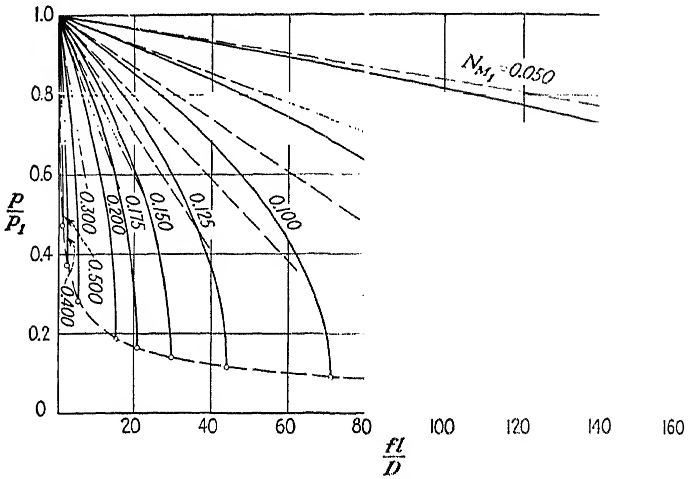


FIG. 252.—Variation of pressure ratio with Mach's number and fl/D for an insulated pipe.

lb. and $V_0 = 1404$ ft. per sec. The final temperature of the gas, as obtained from the expression $pv = RT$, is

$$T_0 = 470°\text{F. abs.} = 10.6°\text{F.} = -12.2°\text{C.}$$

In order to determine the friction factor f, the value of the Reynolds' number must be found. This in turn necessitates a knowledge of the coefficient of viscosity of methane for the range of temperatures involved. Such data[1] are given in the following table.

Temperature, °C.	−181.6	−78.5	0	17	100
$\mu \times 10^6$, poises	34.8	76.0	102.4	108.5	135.2

[1] "International Critical Tables," vol. 5, p. 3, McGraw-Hill Book Company, Inc., New York, 1926–1930.

The values of the coefficient of viscosity corresponding to the initial and final temperatures are then $\mu_1 = 115.3 \times 10^{-6}$ and $\mu_0 = 98.9 \times 10^{-6}$ poises, or in English units 2.40×10^{-7} and 2.06×10^{-7} slugs per ft. sec., respectively. The initial and final values of the Reynolds' number are therefore

$$N_{R_1} = \frac{\rho V_1 d}{\mu_1} = \frac{V_1 d}{g\mu_1 v_1} = \frac{113.5 \times 0.5}{32.2 \times 2.40 \times 10^{-7} \times 2.495} = 2.95 \times 10^6$$

and

$$N_{R_0} = \frac{V_0 d}{g\mu_0 v_0} = \frac{1404 \times 0.5}{32.2 \times 2.06 \times 10^{-7} \times 30.90} = 3.41 \times 10^6$$

Then for this pipe an average value of the friction factor would be about 0.017 (see Fig. 140, page 205). The curves

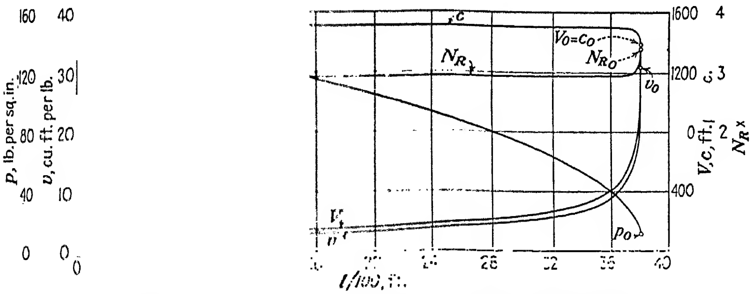


FIG. 253. Conditions in the interior of an insulated pipe.

of Figs. 251 and 252 for $N_{M_1} = 0.075$ are now used to determine the variation of pressure, specific volume and velocity, the results being shown graphically in Fig. 253. The limiting value of fl/D is 129.6 so that the length of the pipe becomes 3810 ft. The values of the acoustic velocity as determined by the formula $c = \sqrt{kgpv}$ are also plotted in this figure and it appears that the velocity of the gas increases with the distance from the inlet, while the acoustic velocity decreases slightly, both velocities being the same at the outlet. From the values of pressure and specific volume along the pipe the corresponding temperatures may be determined. The coefficients of viscosity and the Reynolds' numbers for these points are then easily found and a curve showing the variation of N_R with distance down the pipe is also included in Fig. 253. This curve shows that the assumption

of a constant Reynolds' number and friction factor is not far from the truth in this particular problem.

If the back pressure on the exit of the pipe were made less than the limiting value of 10.42 lb. per sq. in., there would be no increase in the rate of discharge. When the back pressure is increased, the length of pipe must be decreased if the initial conditions, pipe diameter, friction factor and discharge are to remain unchanged. An increase in the length of the pipe beyond the limiting value of 3810 ft. would require a change in the value of N_{M_1} and therefore in the weight W flowing per second.

198. Comparison of Compressible and Incompressible Fluid Flow Theories.—It is of considerable interest to compare the results obtained for the flow of gases in insulated pipes with those given in Chap. IX based on the assumption of an incompressible fluid. The equation for the loss in pressure in a pipe of length l under the latter conditions is

$$p_1 - p = wh = \frac{h}{v} = \frac{flV^2}{2Dgv} \tag{49}$$

and in the case of an incompressible fluid in steady motion the specific volume v and the velocity V are both constants. If Eq. (49) is divided through by p_1, it may be solved for the ratio p/p_1, which is found to be

$$\frac{p}{p_1} = 1 - \frac{flV^2}{2Dgp_1v} \tag{50}$$

Since the fluid is assumed to be incompressible, the velocity of sound in it should theoretically be infinitely large, but the combination $\sqrt{kgp_1v}$ may still be considered as numerically equal to the acoustic velocity c_1, which the fluid would have if it were compressible. Equation (50) may therefore be written in the form

$$\frac{p}{p_1} = 1 - \frac{k}{2}\left(\frac{V}{c_1}\right)^2 \frac{fl}{D} = 1 - \frac{k}{2}(N_{M_1})^2 \frac{fl}{D} \tag{51}$$

so that, as in the case of gas flow, the pressure drop depends on the ratio of the inlet velocity to the velocity of sound and the product of the friction factor by the length-diameter ratio. For a given value of N_{M_1} it appears that the pressure ratio is a linear function of the quantity fl/D. The curves showing this ratio as a function of fl/D for different values of N_{M_1} are drawn in Fig.

252 with dashed lines and it appears that they are tangent to the exact curves at the point where fl/D equals zero. As the distance down the pipe or as the value of fl/D is increased, the difference between the results of the two methods becomes more and more pronounced. Thus, for example, in the case where $N_{M_1} = 0.075$, the error in the pressure ratio made by assuming an incompressible fluid is 10 per cent when $fl/D = 75$. The limiting value of fl/D in this case is 129.4 so that the error is less than 10 per cent only for about the first quarter of the pipe. The theory of incompressible fluids gives no information about the existence of limiting pressures and values of fl/D and, even for low values of Mach's number, the assumption of incompressibility is satisfactory only for a limited length near the inlet. Therefore, as in the discussion of the Venturi meter in Art. 184, it is recommended in connection with the flow of gases in pipes that compressibility should be taken into account unless definite figures are available which show that the assumption of an incompressible fluid does not involve any serious errors.

The theory discussed in the preceding articles has many important applications such as in the design of long natural-gas lines, municipal gas-supply systems, air ducts for heating and ventilating systems and various industrial installations where gases are piped from one point to another. The theory is applicable to steam insofar as it may be regarded as a gas that obeys the laws stated above. Charts similar to those of Figs. 250, 251, and 252 for air ($k = 1.406$) will be found in the technical literature, along with further advanced thermodynamic treatments of these problems.[1]

Problem 383. At the entrance to a long insulated pipe line carrying methane, the pressure is 120 lb. per sq. in. abs., the temperature is 107.6°F. and the velocity is 95 ft. per sec. The pipe is smooth steel and has a diameter of 4 in. (*a*) Determine the limiting values of the pressure, flow velocity and acoustic velocity at the exit. (*b*) What is the length of the pipe? (*c*) Determine the variation in pressure, velocity, acoustic velocity and specific volume with distance from the pipe inlet. Plot the results in graphical form.

[1] SCHÜLE, W., "Technical Thermodynamics," pp. 274–316, Sir Isaac Pitman & Sons, London, 1933.

STODOLA, *op. cit.*

CHAPTER XV

DYNAMIC SIMILARITY

199. Experiments in Fluid Mechanics.—In discussing the problems of fluid mechanics in the preceding chapters an effort has been made to develop a rational theory for the flow involved in each case. Such a theory, if complete and entirely correct, would express both qualitatively and quantitatively the relations between all the factors that affect flow in each case. A survey of the problems discussed will bring out the fact that in most cases such theories as have been developed do not yield completely quantitative solutions and it has frequently been necessary to turn to experimental research for workable results. Such research may have been that required to furnish constants to serve as coefficients and exponents with which to modify the theory developed, or it may have been research work which gives knowledge of the effect of one or more of the various factors involved in the flow. Even where quantitative theories have been worked out, it is essential that they be verified experimentally because such theories are usually based on certain simplifying assumptions that are not entirely satisfied by real fluids.

Much of the experimental information given in this text was obtained by full-scale work. This is largely true, for example, of the work on pipes, weirs, orifices and channels. However, a large part of the advance in fluid mechanics, and in engineering work related to it, is due to the practice of making observations and tests on small-scale models. The resistance and behavior of an airplane are determined by experiments in air with a model geometrically similar to the actual airplane but having a linear scale which may be as small as one-twentieth of the prototype. Likewise the power requirements of a boat are predicted from towing tests on models only a few feet long. Hydraulic developments, river improvements and harbor works are studied in detail on small-scale models, and the performances of propellers, turbines and pumps are predicted on the basis of tests on small homologous machines.

Because of the broad field of application of model experimentation in fluid mechanics, no discussion of that subject can be regarded as complete without some study of the fundamental principles on which such tests are based. When a test is made on a geometrically similar model of a certain device or arrangement, the first question that naturally arises is concerned with the relation between the forces that act on the model and the corresponding ones involved in the fluid motion connected with the full-scale prototype. Such a question immediately introduces the subject of dynamic similarity or similitude which has been touched on to some extent in the preceding chapters.

200. Dynamic Similitude. In general it may be said that two fluid motions are dynamically similar when, first, the boundaries of the flows are geometrically similar, and, second, when the corresponding streamlines are likewise similar in shape. This latter condition requires that the various forces acting on corresponding fluid elements of the two systems must have the same ratio to one another for both flows.

The first step in determining the analytical condition for dynamic similarity of a given fluid flow with that of its prototype is to determine what forces are involved in these motions. If all the properties possessed by real fluids, such as viscosity, inertia and compressibility, are considered simultaneously, the results obtained are usually so complicated in form that it is difficult to bring out clearly the significance of the most important factors. Therefore the usual procedure is to introduce certain simplifying assumptions so that the influence of the different items involved can be studied separately. In the most general type of fluid motion conceivable there might be as many as five different kinds of forces acting on the elements of the fluid. These forces are as follows:

1. Pressure forces.
2. Inertia forces.
3. External forces such as those due to the attraction of gravity.
4. Viscous forces.
5. Elastic forces.

In some cases there are also forces due to the presence of a surface film but such forces are considered to be negligible here. However, they may be very important in some problems, for

example, in dealing with capillary waves, flow with a free surface at low velocity or the behavior of small jets at low heads.

In any fluid flow the pressure forces acting may be considered as being of two kinds: first, those produced by hydrostatic action; second, those which result directly from the motion of the fluid. This study of similitude is concerned only with the latter variety of pressure force.

In addition to giving information as to the nature of each of these individual forces, an analytical study of the fluid motion usually yields an equation of equilibrium relating them with one another. By means of this equation it is possible to express one of the five forces listed above in terms of the other four and thereby reduce the number that must be taken into consideration in establishing the condition of dynamic similarity. The majority of the problems in fluid mechanics involve only three of the above forces and divide themselves into three classes shown in the table below. Each class involves both inertia and pressure

Force Combination	Significant Ratio
1. Viscous, inertia, pressure forces	$\dfrac{\text{Inertia force}}{\text{Viscous force}}$
2. Gravity, inertia, pressure forces	$\dfrac{\text{Inertia force}}{\text{Gravity force}}$
3. Elastic, inertia, pressure forces	$\dfrac{\text{Inertia force}}{\text{Elastic force}}$

force along with either viscous, gravity or elastic force. The pressure force in each case is provided for by the equilibrium equation and the quotient of the other two becomes the significant ratio, which must remain constant if dynamic similarity is to be realized. In the following pages separate articles will be devoted to each of the tabulated force combinations and each significant ratio will be evaluated. Other combinations will also be considered.

In most cases the derivation of the differential equation involving the pressure force is beyond the scope of the present work. However, if the kinds of forces to be considered are known, the correct result can usually be obtained. Unless the proper differential equation is derived or the methods of dimensional analysis previously employed are used on the problem, there is no indication from the result obtained that it is the only

combination of the fundamental quantities involved which must be constant to indicate similarity of flow.

201. Dynamic Similarity of Viscous-fluid Motions.—The condition for dynamic similarity of two fluid flows for which the effects of viscosity, pressure and inertia are considered may be determined by making use of the fact that the ratio between the inertia force and viscous force for the two motions must be the same at corresponding points. Pressure forces are also present but it is not necessary to consider these separately since they will be automatically taken care of by application of the condition of equilibrium to the entire force system.

Now the inertia force acting on an element of fluid is equal to the mass of the element multiplied by its acceleration. The mass is equal to the mass density ρ times the volume, which may be regarded as proportional to the cube of some characteristic length l. The mass is then proportional to ρl^3. The acceleration is the time rate of change of velocity and may therefore be expressed as proportional to the velocity divided by some convenient period of time, that is, V/t. The time may also be written as proportional to the characteristic length l divided by the velocity so that finally the acceleration is proportional to V^2/l. Then the inertia force is proportional to $\rho l^3 V^2/l$ or $\rho l^2 V^2$.

The viscous force is the product of shear stress and an area. It is known that the shear stress due to viscosity is $\tau = \mu(dV/dy)$, which is proportional to $\mu V/l$. Then the viscous force is proportional to $\frac{\mu V}{l} l^2$ or to $\mu V l$.

It now appears that the ratio of inertia force to viscous force is proportional to the quantity $\rho V l/\mu$ because

$$\frac{\text{Inertia force}}{\text{Viscous force}} \propto \frac{\rho l^2 V^2}{\mu V l} \propto \frac{\rho V l}{\mu} \tag{1}$$

This expression will be recognized at once as the quantity that has heretofore been known as the Reynolds' number N_R. The important result of the above analysis is that the Reynolds' number is a nondimensional quantity which is proportional to the ratio between the inertia forces and viscous forces involved in the motion of the fluid. The condition for dynamic similarity of two flows past geometrically similar boundaries requires that the ratio of inertia and viscous forces at corresponding

points shall be a constant. It has been shown that this force ratio is proportional to the Reynolds' number so that the condition for dynamic similarity is satisfied when the Reynolds' numbers of the two flows, based on corresponding characteristic lengths and velocities, have the same value. Examples of types of flow to which this treatment is applicable are the turbulent pipe flows of Chap. IX and the flows past submerged bodies, studied in Chap. XII.

It was mentioned at the beginning of this discussion that the ratios involving the pressure force would be automatically taken care of by means of the equation of equilibrium. This statement may be readily verified without actually setting up that equation. Let two fluid motions for which dynamic similarity is desired be distinguished by the subscripts 1 and 2, and let it be assumed that the following relationships exist between the quantities that characterize these flows:

$$\begin{aligned}
&\text{Density:} && \rho_1 = a\rho_2 \\
&\text{Velocity:} && V_1 = bV_2 \\
&\text{Length:} && l_1 = cl_2 \\
&\text{Viscosity:} && \mu_1 = d\mu_2 \\
&\text{Pressure:} && p_1 = ep_2
\end{aligned}$$

The terms a, b, c, d and e are constants for two points which are in geometrically similar locations but they may be different for each pair of such points. The pressures considered here, as pointed out previously, are only those due to the motion of the fluids.

If the Reynolds' number for these flows are equal, then

$$\frac{\rho_1 V_1 l_1}{\mu_1} = \frac{\rho_2 V_2 l_2}{\mu_2}$$

and, on substituting the above relations in the left-hand expression, it follows that

$$\frac{abc}{d} = 1 \tag{2}$$

The pressure force, which has not been considered previously, is equal to the pressure multiplied by an area and is therefore proportional to pl^2. Hence the ratio between the inertia and pressure force for flow 1 is proportional to $\rho_1 V_1^2 l_1^2 / p_1 l_1^2$ or

$\rho_1 V_1^2/p_1$. In terms of the corresponding quantities for flow 2, this ratio is $\frac{ab^2}{e}\frac{\rho_2 V_2^2}{p_2}$. If the ratio $\rho V^2/p$ is to be the same at corresponding points, that is, if

$$\frac{\rho_1 V_1^2}{p_1} = \frac{\rho_2 V_2^2}{p_2}$$

then

$$\frac{ab^2}{e} = 1 \tag{3}$$

The ratio between the viscous and pressure force is proportional to $\mu_1 V_1 l_1/p_1 l_1^2$ or $\mu_1 V_1/p_1 l_1$ for flow 1. This may also be written in the form $\frac{bd}{ce}\frac{\mu_2 V_2}{p_2 l_2}$. Equality of this force ratio for flows 1 and 2 requires that

$$\frac{\mu_1 V_1}{p_1 l_1} = \frac{\mu_2 V_2}{p_2 l_2}$$

or that

$$\frac{bd}{ce} = 1 \tag{4}$$

The problem is now to show that, if the Reynolds' numbers of the two flows are equal and Eq. (2) is satisfied, then Eqs. (3) and (4) are also correct and the ratios involving the pressure force are equal at corresponding points. If the value of e from Eq. (3) is substituted in Eq. (4), the result is

$$\frac{bd}{cab^2} = \frac{d}{abc}$$

which is exactly the reciprocal of the value given in Eq. (2). Thus if Eqs. (2) and (3) are true it follows that Eq. (4) is correct. The correctness of the first equation was part of the original hypothesis, that is, that the Reynolds' numbers were made equal. Equation (3) may be satisfied by a proper choice of the reference pressures with respect to which p_1 and p_2 are measured, since neither flow will be modified by the addition of a constant pressure acting throughout the entire mass of fluid. This arbitrary choice of the pressure ratio is further justified by the fact that, if the differential equation of equilibrium for a flow were set up

and integrated, the pressure would be determined except for a constant of integration, which may be given any convenient value.

It has now been shown that, if the Reynolds' numbers of the two flows are equal, then the ratio of inertia and viscous forces is a constant, that is,

$$\frac{\text{Inertia force}}{\text{Viscous force}} = K_1 \tag{5}$$

Furthermore, by proper choice of the arbitrary reference pressures, the ratios of inertia and pressure forces may be made equal so that

$$\frac{\text{Inertia force}}{\text{Pressure force}} = K_2 \tag{6}$$

The combination of Eqs. (5) and (6) shows at once that the third significant ratio is

$$\frac{\text{Viscous force}}{\text{Pressure force}} = \frac{K_2}{K_1}$$

which is also a constant. The condition of dynamic similarity is then completely satisfied by making the Reynolds' numbers equal at corresponding points.

202. Application of Reynolds' Number.—An interesting insight into the significance of the Reynolds' number in problems of fluid resistance may be obtained by considering two limiting cases, the viscous forces predominating in one and the inertia forces in the other. Such conditions would correspond to flows in which the Reynolds' number becomes either vanishingly small or approaches infinity. In the case of a submerged body moving through a fluid where the viscous forces are so great that inertia effects may be neglected, the resistance or drag is proportional to the viscous force and the drag may be written in the form

$$D = k_1 \mu V l \tag{7a}$$

or, after multiplying and dividing by $\rho V l$,

$$D = k_1 \rho V^2 l^2 \left(\frac{\mu}{\rho V l}\right)$$

Putting $\rho V l/\mu = N_R$, replacing $k_1 l^2$ by $k_1' A$, in which A is an area, and finally putting $k_1' = C_{D_1}/2$, this becomes

$$D = C_{D_1}\frac{\rho V^2}{2}\frac{A}{N_R} \tag{7b}$$

which is the same form of expression as Stokes' law for the resistance of spheres (page 306). This is to be expected because this law was derived by Stokes on the assumption that inertia forces are negligible.

When the viscous forces are negligible, that is, when the Reynolds' number becomes very large, the drag of a submerged body is proportional to the inertia force and may be written in the form

$$D = k_2\rho l^2 V^2 \tag{8a}$$

$$= C_{D_2}\frac{\rho V^2}{2} A \tag{8b}$$

which is Newton's law for the resistance of an object, derived previously from considerations of changes in the momentum of the fluid [see Eq. (6), page 302].

It appears from the foregoing that problems in fluid resistance may be classified according to the magnitude of the Reynolds' number. It was shown in the earlier chapters that Reynolds' number is a criterion for resistance to flow of a fluid or motion through a fluid when both viscous and inertia forces are involved. In dealing with flow around immersed bodies, small values of this quantity represent flows in which the drag is proportional to the first power of the velocity, independent of the density, and is expressed in the same form as Stokes' law, while with large values of N_R the drag formula follows Newton's law and the force is proportional to the square of the velocity and is independent of the viscosity. Likewise in dealing with flow in pipes it was seen that with values of N_R less than critical the flow is viscous and the pressure drop is proportional to the velocity, and that for very large values of N_R the loss is proportional to V^2 because inertia forces predominate and the curve in the Stanton diagram becomes nearly horizontal. For intermediate values of N_R both types of forces influence the flow and the loss varies with some power of V less than V^2, as indicated by the variable f.

203. Dynamic Similarity of Flow with Gravity Forces Acting.—The flow to be considered next in this discussion of similitude is that in which the forces present are those of pressure, inertia

and the external forces due to the attraction of gravity. The motion is of this type when a free surface is present or when two fluids that do not mix are involved. Examples are the flow in open channels affected by waves and the wave motion caused by the movement of a ship.

The condition for dynamic similarity of flow in these problems may be satisfied by maintaining a constant ratio of inertia and gravity forces at corresponding points. As in the treatment of viscous fluids, the pressure forces will be automatically taken care of by means of the equation of equilibrium. The inertia force, as shown in Art. 201, is proportional to the product ρl^2V^2, where ρ is the mass density of the fluid, l is a characteristic length and V is the velocity. The gravity force is proportional to the density ρ, to the volume of the fluid element, which in turn is proportional to l^3, and to the acceleration of gravity g, so that finally this force is proportional to ρl^3g. The ratio of these two forces is therefore proportional to the expression V^2/lg because

$$\frac{\text{Inertia force}}{\text{Gravity force}} \propto \frac{\rho l^2V^2}{\rho l^3g} \propto \frac{V^2}{lg} \tag{9}$$

In practice it is more convenient to use $V/\sqrt{lg}$, the square root of this ratio. This is permissible because, if this latter value is a constant, then V^2/lg is likewise a constant. A comparison with the study of Art. 176, which was also based on dimensional analysis, shows that the ratio $V/\sqrt{lg}$ is precisely the quantity known as Froude's number N_F. Similitude between flows in which inertia and gravity forces are involved is therefore obtained by requiring that the values of the Froude's number for the different motions, based on corresponding velocities and lengths, shall all be the same. The boundaries for the flows must, of course, be geometrically similar.

The gravity force is significant in any problem involving a wave motion at a free surface. For example, Froude's number, as pointed out in Art. 177, is a significant parameter in formulas for the resistance of ships due to the formation of surface waves. It may also appear in dealing with orifices, weirs and cavitation.

204. Dynamic Similarity for Flow of Elastic Fluids.—Of the three force combinations mentioned in Art. 200 there now remains to be considered only the case in which elastic forces due to the compressibility of the fluid are acting with the pressure and

inertia forces. The ratio on which dynamic similarity depends in a flow of this kind is now the quotient of the inertia force divided by the elastic force. The elastic force is proportional to the bulk modulus of elasticity, E, which is a stress, and to the area on which the force acts, and is therefore proportional to El^2. Hence the ratio of inertia to elastic force is proportional to the quantity $\rho V^2/E$ because

$$\frac{\text{Inertia force}}{\text{Elastic force}} \propto \frac{\rho l^2 V^2}{El^2} \propto \frac{\rho V^2}{E} \tag{10}$$

If the fluids under consideration are gases, then as shown in Art. 180 the bulk modulus E is equal to $\rho dp/d\rho$. Furthermore, if the gases follow the adiabatic law, p/ρ^k = constant, it is necessary that the exponent k be the same for the gases in order that complete similitude may be obtained. That this is the case may be shown by a little more detailed study of the problem. Let the flows of two elastic fluids which are under consideration be characterized by the subscripts 1 and 2 and let the following relationships exist between the various significant quantities:

Density:	$\rho_1 = a\rho_2$
Velocity:	$V_1 = bV_2$
Length:	$l_1 = cl_2$
Bulk modulus:	$E_1 = dE_2$
Pressure:	$p_1 = ep_2$

As in the discussion of similarity for viscous fluids, the pressures considered here are only those caused by the motions of the fluids.

If the ratio of inertia to elastic forces is the same for the two flows, then from Eq. (10)

$$\frac{\rho_1 V_1^2}{E_1} = \frac{\rho_2 V_2^2}{E_2}$$

If ρ_1, V_1 and E_1 are expressed in terms of the corresponding quantities for flow 2, it follows that

$$\frac{ab^2}{d} = 1 \tag{11}$$

The ratio of inertia to pressure force is the same for the two flows if, as shown by Eq. (3),

$$\frac{ab^2}{} = 1 \tag{12}$$

The ratio of pressure force to elastic force is

$$\frac{\text{Pressure force}}{\text{Elastic force}} \propto \frac{pl^2}{El^2} \propto \frac{p}{E}$$

and for similitude $p_1/E_1 = p_2/E_2$, from which

$$\frac{e}{d} = 1 \tag{13}$$

Equation (12) may be satisfied by proper choice of the constant e, this being permitted because of the fact that the base pressures to which p_1 and p_2 are referred may be changed without modification of the flows. If Eq. (11) is satisfied, it follows at once by division of Eq. (11) by Eq. (12) that Eq. (13) is also true. It is necessary to point out, however, that, under the assumption of adiabatic behavior of the gases, the bulk moduli and the pressures are interrelated. If the adiabatic laws for the two cases are

$$\frac{p_1}{\rho_1^{k_1}} = C_1'$$

and

$$\frac{p_2}{\rho_2^{k_2}} = C_2'$$

then, as shown in Art. 180, the corresponding values of the bulk moduli are

$$E_1 = \rho_1 \frac{dp_1}{d\rho_1} = k_1 p_1$$

and

$$E_2 = \rho_2 \frac{dp_2}{d\rho_2} = k_2 p_2$$

It has already been stated in obtaining Eq. (13) that, for similitude, the ratio of pressure force to elastic force must be constant. Hence

$$\frac{p_1}{E_1} = \frac{p_2}{E_2}$$

On introducing the above values of E_1 and E_2, it follows at once that

$$k_1 = k_2 \tag{14}$$

In the cases of gases which behave adiabatically and also satisfy Eq. (14), the expression given earlier in this article as proportional to the ratio of inertia and elastic forces, Eq. (10), may be put in a simpler form. It was shown in Art. 180, Eq. (8), that the velocity of sound in a gas is $c = \sqrt{E/\rho}$. The ratio of inertia and elastic forces is therefore

$$\frac{\text{Inertia force}}{\text{Elastic force}} \propto \frac{\rho V^2}{E} \propto \frac{V^2}{c^2} \tag{15}$$

Thus for geometrically similar boundaries dynamic similarity of two gas flows will be obtained when V^2/c^2 is a constant for corresponding points in the two flow fields. This condition may be replaced by the simpler one that V/c must be a constant, and it will be recalled from the discussions of Chaps. XIII and XIV that this latter ratio is Mach's number, N_M, which has been shown to be of great significance in connection with the flow of compressible fluids. It should be kept in mind that in general c represents the local value of the acoustic velocity corresponding to the values of the pressure and density existing at the point where V is measured.

205. The Pi Theorem. Dimensional Considerations of Orifice Flow.—In the three force combinations outlined in Art. 200 and the discussion of them in the following articles there appears only one significant ratio in each case. These are N_R, N_F and N_M. In many cases more than one of these ratios may appear and other dimensionless ratios may also be introduced.

Suppose that a given flow involves n quantities, the relation of which can be expressed by an exponential equation, and that the number of fundamental units, M, L and T contained in all these quantities is p. It can then be shown that the equation giving the relation between the quantities will contain $n - p$ dimensionless ratios such as N_R, N_F, N_M or others. This statement is known as the Pi theorem.

The orifice will be used as a demonstration but not as a proof of the Pi theorem.[1] Suppose that Q, the discharge of an orifice, depends on the viscosity μ, the density ρ, the linear dimension l of the orifice, the head h and gravity g. Assume that Q can be expressed exponentially as

$$Q = K\mu^a \rho^b l^c h^d g^e \tag{16}$$

In fundamental units this becomes

$$\frac{L^3}{T} = \left(\frac{M}{LT}\right)^a \left(\frac{M}{L^3}\right)^b L^c L^d \left(\frac{L}{T^2}\right)^e \tag{17}$$

from which the following equations are obtained:

$$\begin{aligned} 0 &= a + b \\ 3 &= -a - 3b + c + d + e \\ -1 &= -a - 2e \end{aligned}$$

Values of b, c and e from these equations in terms of a and d are

$$b = -a \qquad e = \frac{1 - a}{2} \qquad c = \frac{5}{2} - \frac{3a}{2} - d$$

Inserting these values in Eq. (16) gives

$$Q = K\mu^a \rho^{-a} l^{\left(\frac{5}{2} - \frac{3a}{2} - d\right)} h^d g^{\frac{1-a}{2}} \tag{18}$$

By rearranging terms this can be written

$$\left(\frac{Q}{l^{\frac{5}{2}}\sqrt{g}}\right) = K\left(\frac{\mu}{\rho l}\,\frac{1}{\sqrt{lg}}\right)^a \left(\frac{h}{l}\right)^d \tag{19}$$

In this equation the three quantities in parentheses are dimensionless ratios. This number was to be expected from the Pi theorem because Eq. (16) contains six variables and Eq. (17) contains three fundamental units so that $n - p = 6 - 3 = 3$. More light may be thrown on the orifice theory by writing Eq. (19) in the form

$$Q = K\,\frac{\sqrt{lg}}{\frac{\rho l}{\mu}}\left(\frac{h}{l}\right)^d l^2\sqrt{gh} \tag{20}$$

[1] For more complete discussion see E. Buckingham, Model Experiments and the Forms of Empirical Equations, *Trans. A.S.M.E.*, vol. 37, p. 263, 1915, or A. C. Chick, "Dimensional Analysis" in "Hydraulic Laboratory Practice," edited by John R. Freeman, p. 787, A.S.M.E., 1929.

The area of the orifice opening is proportional to l^2 and this proportionality constant can be included in a new constant K' along with $1/\sqrt{2}$ and the original K. Then, after inserting some characteristic velocity V in both parts within the first parentheses, the equation for discharge becomes

$$Q = K' \left(\frac{\dfrac{V}{\sqrt{lg}}}{\dfrac{\rho l V}{\mu}} \right)^a \left(\frac{h}{l}\right)^{d-\frac{1}{2}} A\sqrt{2gh} \tag{21}$$

The numerator and denominator of the first ratio as now written are Froude's number and Reynolds' number, respectively, and the ratio may be written N_F/N_R. If the discharge equation for the orifice is now written as

$$Q = CA\sqrt{2gh}$$

and is then compared with Eq. (21), it follows that the coefficient of discharge is

$$C = K'\left(\frac{N_F}{N_R}\right)^a \left(\frac{h}{l}\right)^{d-\frac{1}{2}}$$

The condition for dynamic similarity in orifices is that the dimensionless terms of Eq. (19) must be constant. If the term $\dfrac{\mu}{\rho l}\dfrac{1}{\sqrt{lg}}$ is interpreted as $\dfrac{N_F}{N_R}$, it follows that neither N_F nor N_R need be constant but their ratio must meet that requirement. The requirement that h/l be constant is simply that geometric similarity must exist. This applies not only to the ratio of l and h but to ratios involving every dimension of the tank.

Values of a and d must be determined by experiment. It was shown in Art. 154 that the velocity of a jet from an orifice is independent of μ if $V \propto \sqrt{h}$. If this is strictly true and the contraction is also unaffected by μ, then N_R cannot be a significant ratio and the value of a is zero. The discharge is then also independent of ρ and N_F because they appear in a term affected by the same exponent a.

It is true that C_v and C for sharp-edged orifices are nearly independent of N_R; when they are plotted against N_R, the result is a nearly straight horizontal line. When the orifice is rounded the line is slightly curved and for a tube the curvature becomes

greater. As the side walls of the opening are increased in length and area, the effect of viscosity continues to increase and the curvature of the N_R curve becomes more marked.[1]

206. Application of Dimensional Analysis to Resistance of Floating Bodies.—The resistance of floating bodies was discussed in Art. 177 of Chap. XII. In that article it was pointed out that the drag of a ship hull may be divided into three component parts as follows:

1. Wave-making resistance.
2. Eddy-making resistance.
3. Skin-friction resistance.

The first two of these items are usually combined to form the residuary resistance.

The method for studying ship-hull resistance originated by William Froude is based on the assumption that the expression for the total resistance may be written in the form given by Eq. (60), page 355, that is,

$$D = \frac{\rho V^2}{2}\left| C_f A\, f_1(N_R) + C_r \Delta^{2/3} f_2(N_F) \right. \qquad (22)$$

in which D is the total resistance of the ship hull, ρ is the density of the fluid, V is the velocity of the hull, A is the wetted surface of the hull, Δ is the volume of displacement of the hull, C_f is the coefficient of skin friction based on the wetted surface, C_r is the coefficient of residuary resistance based on $\Delta^{2/3}$, $f_1(N_R)$ is a function of Reynolds' number which determines the relation between the skin-friction drag and N_R and $f_2(N_F)$ is a function of Froude's number which determines the relation between the residuary resistance and N_F.

A rational basis for Eq. (22) is provided by dimensional analysis. Let it be assumed that the resistance of a ship hull is dependent on ρ, V, Δ, g and μ. Then

$$D = f(\rho, V, \Delta, g, \mu)$$

or

$$D = K\rho^a V^b \Delta^c g^d \mu^e \qquad (23)$$

in which K is a nondimensional constant. Inserting the funda-

[1] Typical curves are given by H. Addison, "Applied Hydraulics," p. 62, John Wiley & Sons, Inc., New York, 1934.

mental units M, L and T for the various quantities in Eq. (23), the following relation is obtained:

$$M\frac{L}{T^2} = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b L^{3c} \left(\frac{L}{T^2}\right)^d \left(\frac{M}{LT}\right)^e$$

On equating separately the exponents of mass, length and time on the two sides of this equation, three simultaneous equations are obtained which are

$$\left.\begin{aligned} 1 &= a + e \\ 1 &= -3a + b + 3c + d - e \\ -2 &= -b - 2d - e \end{aligned}\right\} \qquad (24)$$

It will be noted that Eq. (23) contains in all six variables and on applying the Pi theorem, it appears that the result should contain six minus three, or three, nondimensional ratios. Five of the quantities in Eq. (23) are affected by unknown exponents and, since the dimensional method gives only three equations for the determination of these exponents, it is possible to find values of only three of them; these will in general be expressed in terms of the remaining two.

The choice of the exponents to be considered as unknown is completely arbitrary. If d and e are considered as the unknowns, then the solution of Eqs. (24) for a, b and c gives

$$a = 1 - e, \qquad b = 2 - 2d - e, \qquad c = \frac{2}{3} + \frac{d}{3} - \frac{e}{3}$$

The expression for the drag of the hull now becomes

$$D = K\rho^{(1-e)}V^{(2-2d-e)}\Delta^{\left(\frac{2}{3}+\frac{d}{3}-\frac{e}{3}\right)}g^d\mu^e$$

On arranging in separate groups the terms affected by the exponents d and e, the result is

$$D = K\rho V^2\Delta^{2/3}\left(\frac{\Delta^{1/3}g}{V^2}\right)^d\left(\frac{\mu}{\rho V\Delta^{1/3}}\right)^e \qquad (25)$$

The factor Δ in the parentheses may be replaced by a constant times the cube of any convenient linear dimension of the hull, such as its length l. The right side of Eq. (25) may also be

multiplied and divided by 2 and, with the proper change in the coefficient, Eq. (25) becomes

$$D = C_D \frac{\rho V^2}{2} \Delta^{2/3} \left(\frac{\sqrt{lg}}{V} \right)^{2d} \left(\frac{\mu}{\rho V l} \right)^e \tag{26}$$

Reference to the earlier articles of this chapter shows that the quantity $V/\sqrt{lg}$ is Froude's number N_F, while $\rho Vl/\mu$ is Reynolds' number N_R. Previous studies of resistance have shown that when viscosity is an important factor, as in the case of pure skin friction, the drag is a function of Reynolds' number. When gravity forces are involved in the flow, as in the case of wave-making resistance, it was found that the drag was dependent on Froude's number. It might then be expected that, when both kinds of forces are present as in this problem, the resistance should be a function of both Reynolds' and Froude's numbers. This supposition is justified by the result given by Eq. (26).

The three dimensionless ratios which, according to the Pi theorem, are involved in this problem are then N_F, N_R and $\dfrac{D}{\frac{\rho V^2}{2}\Delta^{2/3}}$. The denominator of the last ratio consists of a pressure $\rho V^2/2$ multiplied by the two-thirds power of a volume which is an area. This product then has the dimension of a force so that $\dfrac{D}{\frac{\rho V^2}{2}\Delta^{2/3}}$ is nondimensional.

The methods of dimensional analysis as applied to the ship-hull resistance problem do not lead directly to the result given by Eq. (22). This requires the introduction of Froude's assumption that the resistance may be divided into two parts, residuary and skin-friction drags, and that the first of these may be considered as a function of N_F alone while the second is a function of N_R alone. Equation (26) is then written in the form

$$D = \frac{\rho V^2}{2} \Delta^{2/3} \left[C_f' f_1(N_R) + C_r f_2(N_F) \right]$$

or, putting $C_f'\Delta^{2/3} = C_f A$, where A is the wetted surface,

$$D = \frac{\rho V^2}{2}\left[C_f A\, f_1(N_R) + C_r \Delta^{2/3} f_2(N_F)\right] \tag{27}$$

which is identical with Eq. (22).

Although Eqs. (26) and (27) are different in form, there is a good deal of similarity between them. Both contain Reynolds' and Froude's numbers. The use of Eq. (26) requires that the values of the exponents d and e be known, the former being associated with N_F and the latter with N_R. When Eq. (27) is employed, the functions f_1 and f_2 must be determined, these being associated with N_R and N_F, respectively.

The experimental determination of ship-hull resistance is usually accomplished by towing a small-scale geometrically similar model of the hull in a long channel filled with water. Such a channel is known as a naval tank. In order to have dynamic similarity between the flow around the model and that around the full-scale hull, it is necessary that the Reynolds' and Froude's numbers should have the same values in the two cases. If primed quantities are used to represent the model conditions, then $N_R = N_R'$ and $N_F = N_F'$. It is not difficult to satisfy the condition of equality of Froude's numbers. If the ratio of model and full-scale hull lengths is $l'/l = 1/\lambda$, then the condition that

$$\frac{V}{\sqrt{lg}} = \frac{V'}{\sqrt{l'g}}$$

is satisfied if

$$V' = V\sqrt{\frac{l'}{l}} = \frac{V}{\sqrt{\lambda}} \tag{28}$$

Since λ is generally larger than unity, the proper towing speed of the model is considerably smaller than the speed of the full-scale ship.

If Eq. (28) is satisfied, it is impossible to make the Reynolds' numbers equal. Equality of Reynolds' numbers would require that

$$\frac{\rho V l}{\mu} = \frac{\rho' V' l'}{\mu'}$$

If, as is usually the case, the fluid in the model tank is water, then $\rho = \rho'$ and $\mu = \mu'$ so that

$$Vl = V'l'$$

The speed of the model must then be

$$V' = V\frac{l}{l'} = V\lambda \tag{29}$$

It is obviously impossible to satisfy Eqs. (28) and (29) simultaneously.

In practice, tank tests are made under conditions that give equality of the Froude's number. The skin-friction drag of the model hull is then computed on the assumption that it is equal to the resistance of a parallel flat plate of the same length and wetted surface and having the same Reynolds' number N_R' as the model. The difference between total and skin-friction drags is the residuary resistance and, with its value known, the expression $C_r f_2(N_F)$ in Eq. (27) may be computed. The residuary resistance for the full-scale hull is then determined by using this value in the equation

$$D_{\text{residuary}} = C_r\frac{\rho V^2}{2}\Delta^{2/3} f_2(N_F)$$

The skin-friction drag of the full-size hull is then computed on the same basis as for the model and the sum of this value and the above expression for residuary drag gives the total drag of the full-scale hull.

207. Dimensional Considerations of Resistance of Submerged Bodies.—The resistance of a body submerged in a mass of fluid and in motion relative to the fluid presents another type of problem to which dimensional analysis may be applied with considerable success. In Chap. XII, Art. 158, it was shown that, if the principal forces acting on the fluid elements are pressure, inertia and viscosity, then the resistance may be expressed in the form

$$D = C_D\frac{\rho V^2}{2}A\,f(N_R) \tag{30}$$

in which A is an area of the body, usually that projected in a plane normal to the direction of motion, while the remaining quantities have the same significance as in Art. 206. Equation (30) is applicable to bodies moving through a liquid or a gas, provided in the latter case the velocity is not too great.

The determination of the resistance of an airplane presents some interesting problems, particularly in connection with the use of models. According to the principles of dynamic similitude the model of the airplane should be geometrically similar to its full-scale prototype and the Reynolds' numbers should be the same in both cases. Most aerodynamic experiments of this kind are conducted in the wind tunnel, a device for producing a current of air relative to the model. The wind tunnel usually consists of a long tube constructed somewhat in the form of a

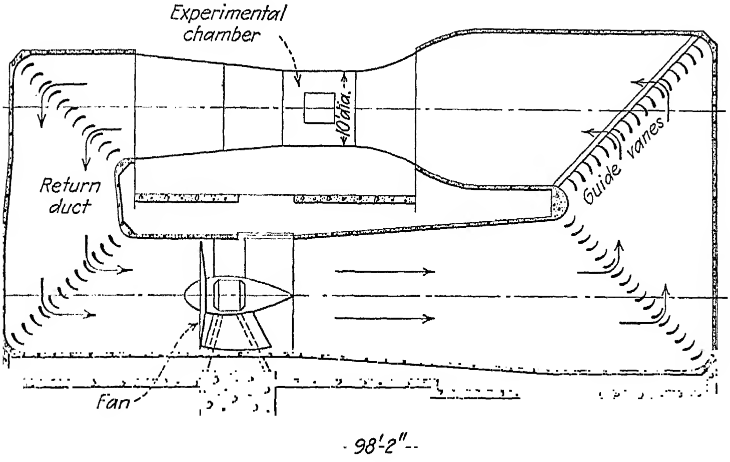


Fig. 254.—Closed-throat type of wind tunnel with return duct—Guggenheim Aeronautics Laboratory of the California Institute of Technology.

Venturi tube. A fan is placed at one end for the purpose of producing motion of the air past the model, which is suspended in the throat in such a way that the forces acting on it can be measured. In Fig. 254 is shown a longitudinal section of the wind tunnel at the California Institute of Technology[1] and in Fig. 255 is found a photograph of the experimental chamber and model setup in the University of Michigan tunnel. These installations are representative of modern equipment of this kind.

The maximum size of a wind-tunnel model is determined by the dimensions of the throat section at which the model is placed. With the exception of a few extremely large tunnels,

[1] Millikan, C. B., and A. L. Klein, The Effect of Turbulence, *Aircraft Eng.*, vol. 5, pp. 169–174, 1933.

most laboratories can test models whose linear dimensions range from one-thirtieth to one-tenth of the dimensions of the full-scale airplanes they represent.

If l and l' denote a certain length on the airplane and its model, respectively, then the scale ratio of the model is $l'/l = 1/\lambda$.

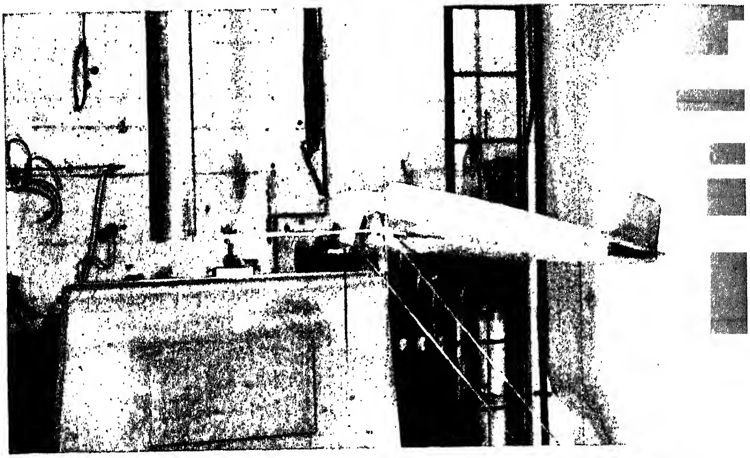


FIG. 255.—Experimental chamber and model setup in the University of Michigan wind tunnel.

Equality of Reynolds' numbers requires that

$$\frac{\rho' V' l'}{\mu'} = \frac{\rho V l}{\mu} \tag{31}$$

or, since the densities and viscosities are usually the same,

$$V'l' = Vl$$

The proper wind speed for a model test is then

$$V' = V\frac{l}{l'} = V\lambda \tag{32}$$

As an example, consider the case of an airplane designed to fly at 200 m.p.h., the model of which has a scale factor $\lambda = 10$. According to Eq. (32) the speed in the wind tunnel should then be

$$V' = 200 \times 10 = 2000 \text{ m.p.h.}$$

This tremendously high velocity is far beyond the capacity of existing wind tunnels. Not only would an enormous amount

of power be required to operate the fan, but it should also be noted that this speed is considerably in excess of that of sound. Thus the flow around the model would be in the range where compressibility effects are important. The Mach's numbers in the two cases would be $N_M' = 2000/762 = 2.63$ for the model and $N_M = 200/762 = 0.263$ for the full-size airplane so that, while similarity of the viscous and inertia forces would have been obtained, the effects of compressibility on the model test results would be so pronounced as to render them practically worthless.

In practice, wind tunnels are usually operated at speeds of the order of 40 to 250 m.p.h. so that on small-scale models equality of Reynolds' numbers cannot be expected. .This does not render the tests worthless because it is possible to extrapolate the results thus obtained to the higher Reynolds' number of the full-scale unit. The basis for such an extrapolation depends on having available experimental or theoretical laws giving the relationship between the forces and the Reynolds' numbers for the complete range between the value obtained in the model test and that corresponding to actual flight. At the present time such information is being obtained from laboratories operating full-scale wind tunnels, by comparison of model tests with measurements of speed and power in flight and by the use of theoretical developments such as the skin-friction drag formulas for flat plates parallel to the airstream.

Another satisfactory method for obtaining the high values of the Reynolds' numbers corresponding to the flight of an actual airplane is the use of a completely closed tunnel employing a gas having a lower kinematic viscosity than that of atmospheric air. An inspection of Eq. (31) shows that, if the value of $\nu = \mu/\rho$ is decreased, then the Reynolds' number for the model test will be increased provided the ratios of velocities and characteristic lengths have already been established. The only application that has been made of this principle up to the present time is to use compressed air as the gas in the tunnel, this idea first being proposed by Munk.[1] The entire tunnel is placed inside a sealed

[1] MUNK, M. M., and E. W. MILLER, The Variable-density Wind Tunnel of the National Advisory Committee for Aeronautics, *NACA Tech. Rept.* 227. See also JACOBS, E. N., and I. H. ABBOTT, The NACA Variable-density Wind Tunnel, *NACA Tech. Rept.* 416.

shell and the pressure is increased to several atmospheres. Since this change occurs at approximately constant temperature, the absolute viscosity μ is unchanged and there is only an increase in the density which is proportional to the increase in pressure. For this reason tunnels of this type are known as variable-density tunnels. This type of tunnel has an advantage over the atmospheric tunnel in that high Reynolds' numbers may be obtained without excessively high speeds. The acoustic velocity in a variable-density tunnel is $c = \sqrt{kp/\rho}$ and, since p and ρ are varied in the same proportions, c remains constant so that undesirable compressibility effects may be avoided. Thus in the case of the model discussed on page 440 a pressure of 10 atmospheres and a tunnel velocity of 200 m.p.h. would give a model Reynolds' number equal to the full-scale value, while the Mach's numbers would be identical. If desired, the pressure might be increased to say 20 atmospheres and a velocity of only 100 m.p.h. employed. Equality of Mach's numbers would no longer be maintained but, since both the model and full-scale values of N_M would be well below unity, no serious difficulties would result.

In 1936 there were only two such tunnels in operation, one at the National Physical Laboratory in England and the other at the Langley Field laboratories of the National Advisory Committee for Aeronautics in the United States. The latter employs pressures up to 20 atmospheres while in the English tunnel pressures as high as 25 atmospheres can be obtained. The sizes of models and maximum air speeds are such that in the English tunnel Reynolds' numbers as high as 6.83×10^6 can be obtained, while in the American tunnel the maximum value is about 3.4×10^6. These values are for airfoil models and are based on the wing chord.

Although the Reynolds' numbers obtained in the variable-density tunnels are not so large as those corresponding to the flight of the fastest and largest airplanes, it is found that the variation of the drag force with Reynolds' number is much more uniform than in the lower ranges where the ordinary atmospheric tunnel operates. Thus the prediction of full-scale force values from such test results should be considerably more accurate than calculations based on the relatively low Reynolds' number of the atmospheric tunnel.

General Problems

384. A spherical particle is dropped into a tank of castor oil and a similar particle is dropped into water. Both liquids are at a temperature of 50°F. What must be the ratio of the diameters of the particles if the flows are dynamically similar at the same velocity?

385. The skin friction drag coefficient of a flat plate parallel to a stream of fluid varies with $(N_R)^{-\frac{1}{5}}$ when the boundary layer is turbulent. How does the drag vary with velocity? Referring to the data on flat plates given in Chap. XII, determine whether or not n can be greater than 2 in the relation $D \propto V^n$.

386. The wind resistance of a tall building is assumed to be dependent on the density and viscosity of the air, the wind velocity and the width and height of the building. Develop an expression for the resistance of the building in terms of these quantities and find the nondimensional ratios involved in the problem.

387. Develop an expression for the resistance of a bridge pier placed in a stream of water of depth d if it is assumed that the viscosity may be neglected. The principal dimensions of the cross section of the pier are a in the direction of motion and b at right angles to it.

388. A 4 in. circular orifice in a smooth 12-in. water pipe has pressures of 40 lb. per sq. in. abs. and 24 lb. per sq. in. abs. on the upstream and downstream sides, respectively. The flow of water through an orifice in a 6-in. line is required to be dynamically similar. Find (a) the ratios of the discharges in the two cases and (b) the ratio of the pressure drops. Are the ratios of the upstream and downstream pressures necessarily the same?

389. Water flows in a rectangular channel 6 ft. deep and 12 ft. wide with a velocity of 4 ft. per sec. What is the flow in a geometrically similar channel of the same material and 1 ft. deep to produce the same shear stress at the walls?

390. Using the Manning formula for velocity in an open channel of rectangular cross section, determine the relation between the slope s' in a model and the slope s in the full scale channel (a) to make the Reynolds' numbers equal, (b) to make the Froude's numbers equal. Assume both channels to be constructed of the same material. The geometric scale ratio is $l'/l = 1:\lambda$.

391. A pipe 6 in. in diameter is rolled from corrugated sheet. Another pipe 12 in. in diameter is rolled from the same sheet. Can the flow of water through these two pipes be made to satisfy the condition of dynamic similarity? Why?

392. In order to make experiments on the flow through a river bed, it is proposed to employ a distorted model having a horizontal scale of $\frac{1}{250}$ of the actual river bed and a vertical scale of $\frac{1}{50}$. Is it possible to obtain dynamic similarity with the full-scale flow under such conditions? Why?

393. The average height of the projections which form the roughness of a pipe wall is ϵ. Develop an expression for the head lost in such a pipe line, considering viscosity but neglecting compressibility. Express the result in a form comparable with the *Darcy formula*.

394. A ship is to operate at 25 m.p.h., the length of the hull being 175 ft. Tests are being made on a geometrically similar model 10 ft. long. At what speed should the model be towed?

395. A ship-hull model is constructed on a $\frac{1}{12}$ scale. The full-size ship is 150 ft. long; it has a wetted surface of 2400 sq. ft. and a volume of displacement of 7350 cu. ft. The total resistance of the model at a Froude's number of 0.25 is 2.2 lb. What is the corresponding speed of the ship and what is its total resistance?

396. Show that the quantity of water flowing over a weir is in general a function of both Reynolds' and Froude's numbers. For what types of weirs is Reynolds' number important?

397. An airplane having a mean wing chord of 12 ft. is designed to fly at 250 m.p.h. It is desired to determine its resistance by tests on a $\frac{1}{12}$ scale model in an atmospheric wind tunnel. What should be the wind velocity in the tunnel? What are the corresponding values of Mach's numbers?

398. What conditions would be required in a variable-density wind tunnel for the test of Prob. 397 if the Reynolds' and Mach's numbers for the model are to be the same as their full-scale values?

399. An airship 475 ft. long is designed to cruise at 90 m.p.h. A model having a scale of $\frac{1}{20}$ is available and it is proposed to make resistance tests in water at 60°F. What velocity of the stream would be required? If the water were heated to 120°F., what would be the required velocity?

400. Show that the thrust of a propeller moving through a gas is dependent on Reynolds' and Mach's numbers and on the advance-diameter ratio.

401. The quantity of water passing through a turbine or centrifugal pump is dependent on the following quantities: ω is the angular velocity of the wheel, H is the head, D is the diameter of the wheel, ρ is the density of the fluid, μ is the viscosity of the fluid and g is the acceleration of gravity. Develop an expression for Q and show that Reynolds' and Froude's numbers based on peripheral velocity of the wheel enter into the problem. What other dimensionless ratios are involved?

CHAPTER XVI

SPECIAL PROBLEMS IN FLUID MECHANICS

208. The Measurement of Viscosity.—In any study of the flow of viscous fluids complete solutions to problems may be obtained only when the viscosity of the fluid is known. The methods employed in scientific and industrial work for the determination of this quantity are therefore worthy of attention. The subject of the measurement of viscosity is known as viscometry and the instruments employed for this purpose are called viscometers or viscosimeters.

The operation of a viscometer usually involves a flow of the fluid of which the viscosity is to be determined. The use of such an instrument then depends on either a theoretical or an empirical knowledge of a quantitative relationship between the viscosity and certain measurable characteristics of the flow. An ideal type of viscometer would be one in which the nature of the flow involved is completely determined by the viscosity of the fluid. By means of an analytical relation between the viscosity and the quantities characterizing this flow it would then be possible to compute the numerical value of either the absolute or the kinematic viscosity coefficient. In the viscometers in actual use this ideal is never completely attained and it is always necessary to introduce certain correction factors or to calibrate the instruments with fluids of known viscosity. However, if the required corrections or calibrations are properly made, it is usually possible to obtain a value of the viscosity expressed in the appropriate fundamental units of mass, length and time.

In some types of viscometers, particularly those used in industrial work, the flow may be only slightly affected by the viscosity, other considerations being much more important. In such cases the correction factors are usually quite large and it is often difficult to find a simple, direct relationship between the fundamental coefficient of viscosity and the quantities measured. For this reason the viscosity measurements are

frequently expressed in units that are peculiar to the viscometer used. Such values may in some cases be suitable for comparative work but they cannot be employed in the problems of the preceding chapters without transformation to the fundamental units of mass, length and time.

209. Transpiration Methods of Viscometry.—Among those methods for the determination of viscosity which rest on a sound scientific basis, the most widely used is undoubtedly the so-called transpiration method. In this method observation is

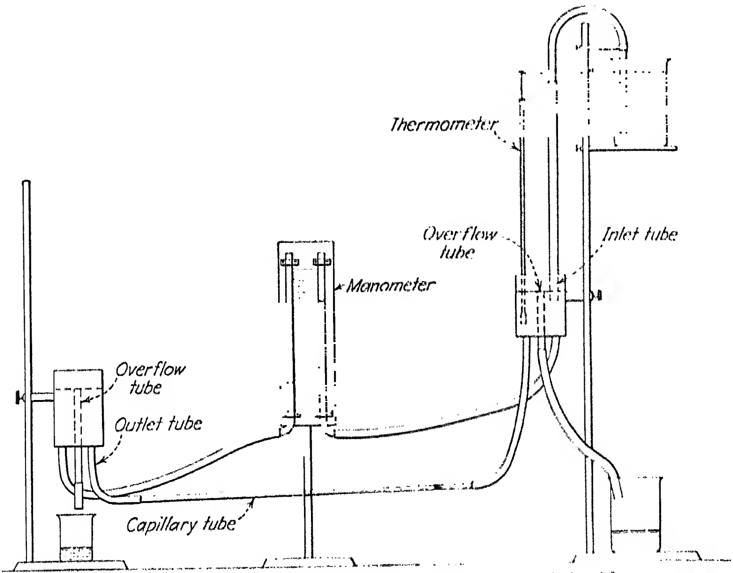


FIG. 256.—Apparatus for measuring viscosity of liquids.

made of the time required for a given amount of fluid to flow through a capillary tube of known diameter and length under a known pressure difference. By selecting a tube of such a size and length that the flow is laminar, the Hagen-Poiseuille law is applicable. This law is expressed by Eq. (16), page 173, which when solved for μ gives

$$\mu = \frac{\pi(p_1 - p_2)d^4}{128Ql}$$

Obviously the method is more suitable for liquids than for gases but special arrangements for timing the flow have been devised so that the viscosity of gases may also be determined by this

method. A simple arrangement of this kind for use with liquids is shown in Fig. 256. This apparatus consists of a capillary tube of known length and diameter connected by rubber tubing to two tanks in which overflow tubes are placed so that the surface level in each of the tanks may be maintained constant. Manometers are provided for measuring the difference in level or head, which, assuming no loss in the relatively large connecting tubes, is all consumed in causing flow through the capillary tube. A thermometer suspended in one of the tanks indicates the temperature of the liquid. The liquid is fed continuously into the higher tank at such a rate that some runs out of the overflow tube. The balance passes through the capillary, out of the overflow tube of the lower tank and into a receptacle placed beneath it. By measuring the quantity accumulated in this receptacle in a given time, it is a simple matter to compute the viscosity of the liquid.

A modification of the transpiration method is employed in the Ostwald viscometer shown in Fig. 257, in which, as before, a known quantity of liquid is timed as it flows through a capillary tube. In this instrument, which is mounted vertically, the flow is caused by hydrostatic pressure of the fluid itself. Thus a known quantity of liquid is placed in the upper bulb A and the time interval required for the meniscus to pass the two marks m_1 and m_2 is noted. Because of the construction of the instrument it is difficult to determine the dimensions of the capillary and consequently this viscometer must be calibrated with some fluid of known viscosity.

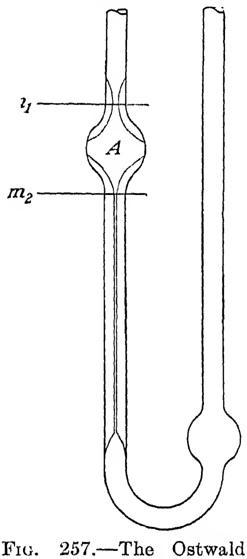


FIG. 257.—The Ostwald viscometer.

Problem 402. A capillary-tube viscometer such as that shown in Fig. 256 is to be used for the determination of the viscosity of water at 59°F. The capillary has a diameter of 0.075 in. and a length of 10 in. What is the transpiration time for 15 cu. in. of water if the difference in head is 5 in. of water?

403. What is the maximum head that can be used on the viscometer of Prob. 402 when determining the viscosity of water at 59°F.?

404. What is the absolute viscosity of an oil if it requires 320 sec. for 8 cu. in. to flow through a capillary-tube viscometer having a diameter of 0.055 in. and a length of 12 in.? The pressure difference is 7.5 in. of water.

210. Other Scientific Viscometry Methods.—The transpiration method is not suitable for very viscous fluids because of the extremely long time required for the flow to take place. For such substances a more satisfactory method is to measure the time of fall through the fluid of a small solid sphere of known weight. The viscosity may then be computed by the application of Stokes' law.

Other scientific methods involve the measurement of the time of damping of the oscillations of a horizontal disk suspended by a wire fastened to its center, or the determination of similar data for a spherical pendulum swinging in the fluid. In still another method the fluid is placed in a cylinder of annular cross section and a coaxial cylindrical shell extends into the fluid. The annular cylinder is rotated at constant speed and a measurement of the torque acting on the shell makes it possible to calculate the viscosity. In all these devices the relation between the quantity measured and the viscosity must be determined either by theoretical means or by calibration with fluids of known viscosity. More complete information in regard to the theory and operation of these and other types of viscometers will be found in the literature.[1]

211. Industrial Viscometers.—In technical or industrial work there is a considerable lack of uniformity in the apparatus and methods used in viscometry. A further difficulty lies in the fact that the results of observations with these instruments are usually expressed in units peculiar to the instrument itself rather than in the fundamental units of mass, length and time. In Germany the Englers viscometer is commonly used; in England, the Redwood; the Barbey Ixometre in France; and the Saybolt in the United States. In most of these devices the principle of operation involves the measurement of the time for a certain quantity of the fluid to flow through an opening, which more often resembles an orifice rather than a capillary tube, or at least is so short

[1] BINGHAM, E. C., "Fluidity and Plasticity," Chaps. IV and V, McGraw-Hill Book Company, Inc., New York, 1922.

HERSCHEL, W. H., Determination of Absolute Viscosity by Short-tube Viscosimeters, *Bur. Standards Tech. Papers* 100.

that the Hagen-Poiseuille law does not strictly apply. Thus the true viscosities of different liquids measured in fundamental units are in many cases not directly proportional to the times of transpiration. It is, however, possible to compute the true viscosity by means of empirical relationships, an example of which is given in the next article.

212. The Saybolt Viscometer.—As mentioned in Art. 211 the Saybolt viscometer is commonly used in the United States, this

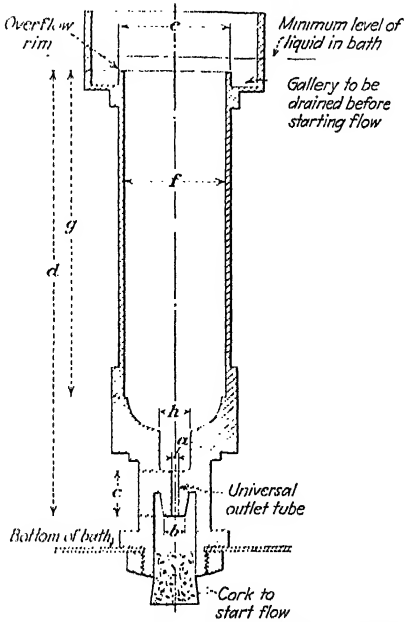


FIG. 258. Oil tube for Saybolt Universal and Furol viscometers.

instrument being the standard prescribed by the American Society for Testing Materials[1] for use in the determination of the viscosity of petroleum products and lubricating oils. There are two Saybolt viscometers employed in this field, both of the same general design but of different dimensions. The Saybolt Universal viscometer is primarily suited for use with lubricating oils, while the Saybolt Furol viscometer is designed for determining the viscosity of heavy fuel oils and liquid asphalt materials such as are employed in road building. In both of these

[1] AMERICAN SOCIETY FOR TESTING MATERIALS, "Standards on Petroleum Products and Lubricants," pp. 304–309, 1935.

instruments the viscosity is measured by the time in seconds required for 60 cc. of the liquid to flow vertically through a capillary tube. If, for example, this quantity of a certain liquid is discharged in 75 seconds, the Saybolt viscosity is said to be 75 seconds. This tube is immersed in a constant-temperature bath and the instrument is provided with heating units so that the viscosity may be determined over a range of temperature. The nature of the tube through which the liquid must flow is shown in Fig. 258. The material to be tested is poured into the large reservoir and is allowed to flow through the outlet tube by removing the cork beneath it. The liquid runs into a glass flask

TABLE XII.—DIMENSIONS OF SAYBOLT OIL TUBES[1]

Dimensions	Saybolt Universal viscometer			Saybolt Furol viscometer		
	Minimum, cm.	Normal, cm.	Maximum, cm.	Minimum, cm.	Normal, cm.	Maximum, cm.
Inside diameter of outlet tube................	0.1750	0.1765	0.1780	0.313	0.315	0.317
Outside diameter of outlet tube at lower end..	0.28	0.30	0.32	0.40	0.43	0.46
Length of outlet tube*..	1.215	1.225	1.235	1.215	1.225	1.235
Height of overflow rim above bottom of outlet tube*............	12.40	12.50	12.60	12.40	12.50	12.60
Outside diameter of overflow rim, at the top*‡...............	†		3.30	†		3.30
Diameter of container*..	2.955	2.975	2.995	2.955	2.975	2.995
Depth of cylindrical part of container*.........	8.8			8.8		
Diameter of container between bottom of cylindrical part of container and top of outlet tube*............	0.9			0.9		

* This dimension is identical in the Saybolt Universal and the Saybolt Furol instruments.
† The minimum value shall preferably not be less than 3.2 cm.
‡ The section of overflow rim shall be bounded by straight lines except that a fillet is permissible at the junction with the bottom of the gallery.

[1] AMERICAN SOCIETY FOR TESTING MATERIALS, *op. cit.*, pp. 304–309.

having a graduation mark indicating a volume of 60 cc. The dimensions of the standard Universal and Furol oil tubes are given in Table XII.

While it is desirable that the viscosity of an oil in Saybolt seconds should be proportional to its viscosity in poises, this is approximately true only for values of the former quantity above 200 sec. The relationship between the kinematic viscosity of an oil measured in centistokes and its transpiration time, t, in Saybolt seconds may be expressed analytically by the following empirical formulas:

$$\nu = 0.226t - \frac{195}{t} \tag{1}$$

for t less than 100 sec. and

$$\nu = 0.220t - \frac{135}{t} \tag{2}$$

for t greater than 100 sec. Either equation is valid when $t = 100$. The Saybolt Universal viscometer may be used satisfactorily for temperatures up to 210°F. provided the oil is not in the neighborhood of either its solid or its flash point. For higher temperatures the Ostwald viscometer immersed in a constant-temperature bath is often employed.

213. Theory of Transpiration-type Viscometers.—In actual use the transpiration method for the determination of viscosity frequently requires the introduction of certain corrections to the basic Hagen-Poiseuille law. This is true in general of all viscometers of this type whether they are used for industrial or for scientific purposes. If the capillary tube of the viscometer is regarded as a circular pipe in which the flow is laminar, then the volume of fluid passing through it in a time t, as was shown in Art. 85, is

$$Q_t = \frac{\pi(p_1 - p_2)d^4t}{128\mu l} \tag{3}$$

Potential energy exists by reason of a pressure difference at the ends of the tube. In the actual flow, however, a certain portion of this energy is transformed into kinetic energy. It is known that the velocity at any point at a distance y from the axis of the pipe, when the laminar flow with its paraboloidal velocity distribution is fully developed, is

$$u = \frac{8V}{d^2}\left(\frac{d^2}{4} - y^2\right) \tag{4}$$

where V is the average velocity. By a method paralleling that used in Art. 101, the kinetic energy of the flow is found to be

$$T = \pi\rho \int_0^{\frac{a}{2}} yu^3\, dy$$

If the value of u from Eq. (4) is substituted in this expression and the integration is carried out, the result is

$$T = \frac{\pi\rho V^3 d^2}{4} \tag{5}$$

This energy contained in the flowing fluid represents a loss in pressure p_T, the magnitude of which may be determined by equating the work done by that pressure in unit time to the kinetic energy T. Thus

$$p_T \frac{\pi d^2}{4} V = T = \frac{\pi\rho V^3 d^2}{4}$$

so that $p_T = \rho V^2$. Putting $V = 4Q_t/\pi d^2 t$, the result is

$$p_T = \frac{16\rho Q_t^2}{\pi^2 d^4 t^2} \tag{6}$$

The pressure difference on the ends of the pipe, effective in overcoming viscous resistance to flow, is therefore

$$p_1 - p_2 - p_T$$

If the pressure difference in Eq. (3) is replaced by this effective pressure difference, then

$$Q_t = \frac{\pi d^4 t}{128\mu l}(p_1 - p_2 - p_T) = \frac{\pi(p_1 - p_2)d^4 t}{128\mu l} - \frac{\rho Q_t^2}{8\pi\mu l t}$$

When this last expression is solved for the absolute coefficient of viscosity, the result obtained is

$$\mu = \frac{\pi(p_1 - p_2)d^4 t}{128 l Q_t} - \frac{\rho Q_t}{8\pi l t} \tag{7}$$

The kinematic viscosity of the fluid is now obtained by dividing Eq. (7) by the density so that

$$\nu = \frac{\pi(p_1 - p_2)d^4t}{128\rho l Q_t} - \frac{Q_t}{8\pi l t} \tag{8}$$

In an actual test measurement will usually be made of the difference in head on the ends of the flow tube rather than the pressure difference. The pressure difference is

$$p_1 - p_2 = w(h_1 - h_2) = \rho g(h_1 - h_2)$$

where w and ρ are, respectively, the specific weight and density. Thus the kinematic viscosity may be expressed in the form

$$\nu = \frac{\pi(h_1 - h_2)gd^4t}{128lQ_t} - \frac{Q_t}{8\pi l t}$$

If the viscometer is designed so that the difference in head is the same for all fluids tested in it, then an expression for kinematic viscosity may be written in the form

$$\nu = At - \frac{B}{t} \tag{9}$$

in which A and B are constants depending on the dimensions and characteristics of the viscometer employed. A comparison of Eqs. (1) and (2) with Eq. (9) shows that they are identical in form. In viscometers of the Saybolt type, the constants A and B also include a correction for the varying head.

Other corrections are required in the use of viscometers of the transpiration type, such as the correction for loss at the inlet of the tube. However, most of them are of the same type as the kinetic energy correction and may be included in the instrument constants. Thus, if the values of A and B are determined by calibration, all the major errors are taken into account.

The constants A and B of Eq. (9) always have positive values so that, for small values of t, ν will have negative values. In this range the correction is more complicated in form and Eq. (9) is incorrect. This difficulty is avoided by the standards set up by the American Society for Testing Materials for the operation of Saybolt viscometers. The Saybolt Universal viscometer is not to be used for times less than 32 sec., while the Furol instrument may not be employed for times less than 25 sec.

Space is not available for a longer treatment of the subject but excellent books are available for the reader who wishes to investigate these problems more fully.[1]

Problem 405. The viscometer shown in Fig. 256 has a capillary tube 0.2 cm. in diameter and discharges 75 cc. of fluid under a head of 15 cm. Determine the constants A and B in Eq. (9) when the tube is (*a*) 1.5 cm. long, (*b*) 12.5 cm. long.

406. The viscosity of a lubricating oil is 175 Saybolt sec. What is its kinematic viscosity?

407. A certain viscometer discharges its standard quantity of castor oil at 15°C. in 300 sec. and at 40°C. in 50 sec. Determine the constants in Eq. (9).

408. The absolute viscosity of an oil having a specific gravity of 0.81 is 1.25×10^{-3} slug per ft.sec. What is its transpiration time when tested in a Saybolt Universal viscometer?

214. Mechanics of Thin Films.—The theory and operation of viscometers as discussed in the preceding articles have for the most part been based on the relations developed for the laminar flow of a viscous fluid in a circular conduit. In the discussion of the resistance experienced by bodies moving through a fluid, as presented in Chap. XII, some attention has been given to the theory of the boundary layer.

There is still another group of problems which is concerned with the behavior of a thin film of viscous fluid in motion between two solid boundaries, and there are a number of applications of this type which are of considerable interest. The basic assumptions on which problems of this group are usually solved are similar to those used in developing the Hagen-Poiseuille law for laminar pipe flow. The thickness of the film is assumed to be very small and the viscosity of the fluid very large; as a consequence of the first assumption, the velocity gradients normal to the bounding surfaces may be of considerable magnitude. Because of these conditions it is concluded that the nature of the flow will be determined primarily by the viscous forces and that the inertia forces may be neglected. Reference to Art. 202 indicates that such problems would therefore correspond to cases where the Reynolds' number of the flow is extremely small.

215. Laminar Flow between Parallel Stationary Plates.—The case of fluid moving between two parallel plane surfaces, both

[1] Cf. footnote, p. 448.

of which are at rest, forms one of the simplest examples of the motion of thin films. This arrangement is shown in Fig. 259, which represents a cross section of the flow taken in the direction of motion. The plates are assumed to be sufficiently large so that the motion may be regarded as two-dimensional. The agency producing the flow may be a pressure difference on the ends of the channel formed by the two surfaces. The origin of a rectangular coordinate system is located at any convenient point half way between the two plates, the x-axis being chosen parallel to the planes of the plates and the y-axis at right angles to them. The distance between the plates is represented by b. Consider a fluid particle in the form of a small rectangular parallelepiped

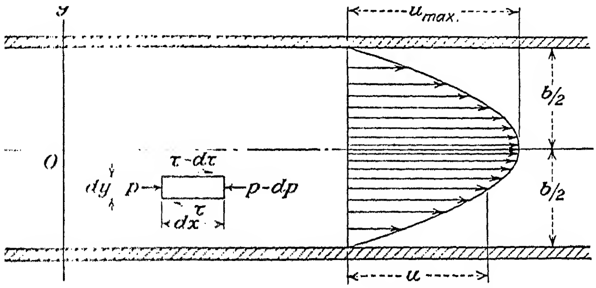


FIG. 259.—Laminar flow between parallel stationary plates.

of unit thickness, with sides parallel to the coordinate axes having lengths equal to dx and dy. If the flow is from left to right, then there will be a shear stress on the lower surface equal to τ, which will act in the negative x-direction. A similar stress on the upper surface acts in the opposite direction and is equal to $\tau - d\tau$. The figure is drawn as though there were a positive velocity gradient in the neighborhood of the particle so that it is moving slightly faster than the one just below it and slower than the one above. This, however, does not affect the generality of the development. The pressures on the left- and right-hand ends of the element may be represented by p and $p - dp$, respectively. The motion is assumed to be steady and independent of x so that no inertia forces are involved and the velocity gradient is a function of y alone. The condition for equilibrium of the element then becomes

$$[p - (p - dp)]dy - [\tau - (\tau - d\tau)]dx = 0$$

or

$$dp\,dy - d\tau\,dx = 0$$

The differential equation for determining the variation of velocity with y is then

$$\frac{dp}{dx} = \frac{d\tau}{dy}$$

or, letting

$$\tau = \mu\frac{du}{dy},$$

$$\frac{dp}{dx} = \mu\frac{d^2u}{dy^2} \tag{10}$$

The solution is obtained by integrating twice with respect to y, the result being

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + A_1y + A_2 \tag{11}$$

The constants of integration, A_1 and A_2, are now determined from the conditions that the velocities are equal to zero at the surfaces of the bounding plates, that is, $u = 0$ for $y = \pm\frac{b}{2}$. The final result is

$$u = -\frac{1}{2\mu}\frac{dp}{dx}\left(\frac{b^2}{4} - y^2\right) \tag{12}$$

As in the case of laminar flow in a pipe, the maximum velocity occurs midway between the bounding surfaces. Here its value is

$$u_m = -\frac{1}{2\mu}\frac{dp}{dx}\frac{b^2}{4} \tag{13}$$

while the velocity distribution is parabolic in form. The value of the average velocity at any section is two-thirds of the maximum, that is,

$$V = -\frac{1}{3\mu}\frac{dp}{dx}\frac{b^2}{4} \tag{14}$$

The pressure decreases linearly in the direction of motion so that the pressure gradient is

$$\frac{dp}{dx} = -\frac{(p_1 - p_2)}{l}$$

where l is the distance between points 1 and 2 at which the pressures are p_1 and p_2. The velocity at any point is then

$$u = \frac{(p_1 - p_2)}{2\mu l}\left(\frac{b^2}{4} - y^2\right) \tag{15}$$

The quantity of fluid passing through a section of unit thickness in the direction normal to the xy-plane is thus equal to

$$Q = 2\int^{\frac{b}{2}} u\,dy = \frac{(p_1 - p_2)b^3}{12\mu l} \tag{16}$$

The two plates of this problem might be considered as the vertical walls of an extremely deep channel of which Fig. 259 represents a horizontal section. If the channel is sufficiently deep so that the free surface at the top and the solid boundary at the bottom do not appreciably affect the flow, then for a depth d the discharge under a pressure difference $p_1 - p_2$ is

$$Q = \frac{(p_1 - p_2)b^3 d}{12\mu l} \tag{17}$$

or for a given discharge the loss in head due to viscosity is

$$h_1 - h_2 = \frac{p_1 - p_2}{w} = \frac{12\mu l Q}{wb^3 d} = \frac{12\nu l Q}{gb^3 d} \tag{18}$$

216. Hele-Shaw's Method for Visualization of Two-dimensional Nonviscous Fluid Motions.—If a solid obstacle in the form of a right cylinder is placed between the two bounding plates with its generators perpendicular to these plates, the problem of Art. 215 becomes considerably more complicated. The presence of the cylinder distorts the flow so that the velocity in any plane parallel to the plates is variable in both magnitude and direction, as shown in Fig. 260. It then becomes necessary to consider velocity components parallel to the x and the z coordinate axes. Both of these velocity components, u and w, are now functions of x and z as well as y. Suppose that the distance between the plates is small in comparison with the dimensions of the body placed between them and that the fluid is so viscous that the inertia forces are negligible in comparison with the shearing forces. It was first pointed out by Sir George

Stokes[1] that under these conditions the streamlines representing the average flow approximate those for the two-dimensional flow of a nonviscous incompressible fluid past an infinitely long cylinder having a cross section identical in size and shape with the obstacle.

This theory has been very successfully applied by Hele-Shaw and others in studying the two-dimensional motion of perfect fluid around various types of cylinders. Hele-Shaw made a

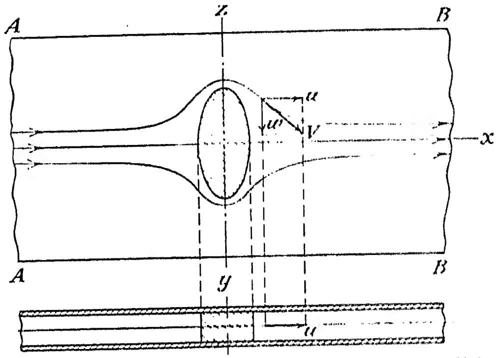


FIG. 260.—Viscous flow past an obstacle between parallel plates.

thin film of glycerin flow between glass plates which were large in comparison with the size of the obstacle placed between them. In his apparatus the sides AB of Fig. 260 are closed and the fluid enters at AA and leaves at BB. The streamlines of the flow are rendered visible by the injection of jets of dye at the upstream opening and the resulting flow pattern may then be studied or photographed through the glass plate.

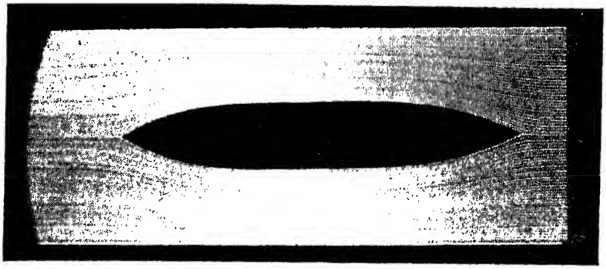
The results of a recent application of this method developed at the University of Liverpool[2] are shown in Fig. 261. The photograph in Fig. 261*a* shows the laminar or streamline flow around a cylinder whose cross section is the water line of a typical ship form. Hele-Shaw also demonstrated that turbulent flow pictures could be obtained with the same device by using

[1] STOKES, G. G., Mathematical Proof of the Identity of the Streamlines Obtained by Means of a Viscous Film with Those of a Perfect Fluid Moving in Two Dimensions, *Math. Phys. Papers*, vol. 5, p. 278.

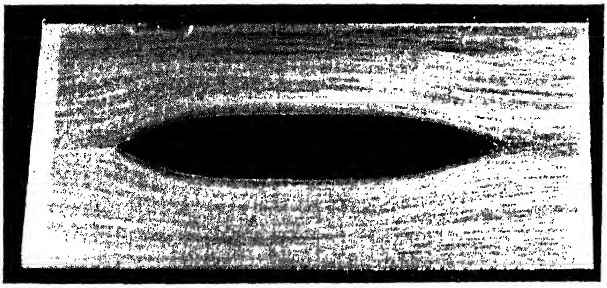
[2] ABELL, T. B., Contribution to the Photographic Study of the Mechanism of the Wake, *Proc. Inst. Naval Architects*, vol. 75, pp. 145–149, 1933.

a larger gap between the plates. A photograph of turbulent flow for the ship hull is shown in Fig. 261*b*.

This method is well adapted to the demonstration and study of two-dimensional nonviscous fluid motions where the boundary conditions as determined by the shape of the obstacle are so complicated as to render mathematical treatment of the problem extremely difficult.



(*a*) Laminar flow.



(*b*) Turbulent flow.

FIG. 261.—Hele-Shaw pictures of flow past a ship-hull section. (*Courtesy of Prof. T. B. Abell.*)

217. The Theory of Lubrication.—Another extremely important application of the hydrodynamic theory of thin films of viscous fluids is found in the study of the lubrication of various types of bearings. While such theory does not completely explain all the phenomena associated with bearing operation, it does serve the function of forming a part of the foundation for a satisfactory theory. The mathematical details of the hydrodynamics of lubrication are in many respects rather complicated and it is proposed to consider here only a relatively simple problem which will serve as an introduction to the more advanced work in this field.

The simplest form of bearing in which the applied load is sustained by a film of viscous fluid is illustrated by the case of a slipper or slide block moving over a plane surface. The problem will first be studied with the assumptions that the surfaces are plane and parallel, that the plate over which the slipper moves is infinite in extent and that the slipper is infinitely long in the horizontal direction normal to its motion. The arrangement is illustrated in Fig. 262. In order to simplify the analysis the principle of relative motion is employed and the slipper is assumed to be stationary in space while the plate moves past it with a velocity equal and opposite to that originally possessed by the slipper. Because the slipper and plate are assumed to be infi-

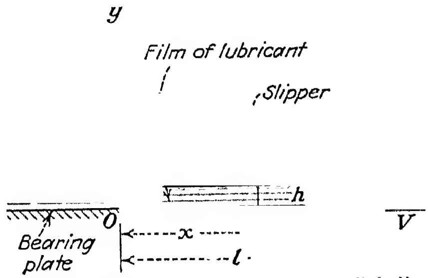


FIG. 262.—Bearing consisting of parallel slipper and plate.

nitely long, the problem becomes a two-dimensional one. Figure 262 represents a cross section normal to the bearing plate and parallel to the direction of motion. The x-axis of a rectangular coordinate system has been chosen in the latter direction and the y-axis normal to it, the origin being located at the left-hand end of the slipper and in the plane of the plate. The width of the slipper is represented by l and the thickness of the film of lubricant, which separates the slipper and plate, by h. Consideration of the forces acting on an element of the viscous film shows that the equation of equilibrium has exactly the same form as that developed in Art. 215 for flow between two stationary plates. It is, of course, necessary that the film be sufficiently thin and viscous so that inertia forces are negligible in comparison with the viscous forces. The basic equations for the solution of this problem are, therefore, the same as Eqs. (10) and (11) on page 456; these are,

$$\frac{dp}{dx} = \frac{d\tau}{dy} = \mu\frac{d^2u}{dy^2} \tag{19}$$

and its integral

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + A_1 y + A_2 \tag{20}$$

In these expressions p is the pressure, τ the shearing stress and u the velocity at any point, while μ is the absolute viscosity and A_1 and A_2 are constants of integration. The determination of the values of these constants depends on the conditions to be satisfied at the boundaries. If the plate is assumed to be moving to the right with a velocity V, then, for $y = 0$, $u = V$, while at the surface of the slipper $y = h$ and $u = 0$. The determination of the values of A_1 and A_2 and their substitution in Eq. (20) give for the velocity at any point in the film the expression

$$u = \left(\frac{1}{2\mu}\frac{dp}{dx}y - \frac{V}{h}\right)(y - h) \tag{21}$$

From the condition of continuity the quantity of fluid passing any cross section of the interspace in unit time is

$$Q = \int_0^h u\,dy = -\frac{1}{12\mu}\frac{dp}{dx}h^3 + \frac{Vh}{2} \tag{22}$$

From Eq. (22) the pressure gradient in the x-direction is

$$\frac{dp}{dx} = \frac{12\mu}{h^3}\left(\frac{Vh}{2} - Q\right) \tag{23}$$

Under the assumptions stated all the terms on the right-hand side of this equation are constants. The pressure at any point in the film is therefore

$$p = \frac{12\mu}{h^3}\left(\frac{Vh}{2} - Q\right)x + B \tag{24}$$

where B is a constant of integration. The value of B may be determined by the fact that, at the left-hand end of the slipper where $x = 0$, the pressure is atmospheric, that is, $p = p_a$. Thus $B = p_a$ and Eq. (24) becomes

$$p = \frac{12\mu}{h^3}\left(\frac{Vh}{2} - Q\right)x + p_a \tag{25}$$

But at the other end of the slipper where $x = l$ the pressure is also $p = p_a$ and this condition can be satisfied only by putting

$\frac{Vh}{2} - Q = 0$. It thus appears that the pressure at every point in the interspace is equal to the atmospheric pressure p_a and, since this pressure acts uniformly on the outer surface of the slipper as well as on the exposed portion of the film, the bearing would not be capable of supporting any load and therefore would be useless in any practical machine.

218. The Inclined-slipper Bearing. It has been shown in the preceding article that a bearing consisting of a parallel slipper and plate is incapable of supporting any load. Some further insight into the physical aspects of the problem may be obtained by considering the velocity distribution across the oil film. In a bearing which is sustaining a load the pressure must increase from atmospheric pressure at the left-hand end of the slipper and return to this value at the right-hand end. Consequently the pressure must reach a maximum value somewhere between these two points as shown in Fig. 263*a*. It is obvious from Eq. (23) that p can vary with x in this manner only when h is a suitable function of x.

The velocity distribution across any section of the oil film is given by Eq. (21), which may be written in the form

$$u_1 + u_2$$

where

$$u_1 = -\frac{1}{2\mu}\frac{dp}{dx}y(h - y)$$

and

$$u_2 = \frac{V}{h}(h - y)$$

The expression for u_1 represents a parabolic velocity distribution with the vertex of the parabola at the point $y = h/2$ and with the parabola opening to the right or to the left, depending on whether dp/dx is positive or negative. Distributions of this type are shown in Fig. 263*b*.

If dp/dx is zero, the velocity is merely u_2, which represents a linear distribution with zero velocity at the slipper surface and a maximum velocity V at the plate. This distribution is shown in Fig. 263*c*.

It is now possible to determine the character of the velocity distribution curves which must accompany the pressure dis-

tribution of the type shown in Fig. 263*a*. At any point the velocity is the combination of the linear arrangement given by u_2 with the parabolic distribution represented by u_1. If p is increasing and dp/dx is positive as between points A and B on the slipper, then the parabola must be subtracted from the straight line and the resulting curve is concave to the right. When dp/dx is negative, the opposite is true and the resultant curve is concave to the left. At the point where p is a maximum,

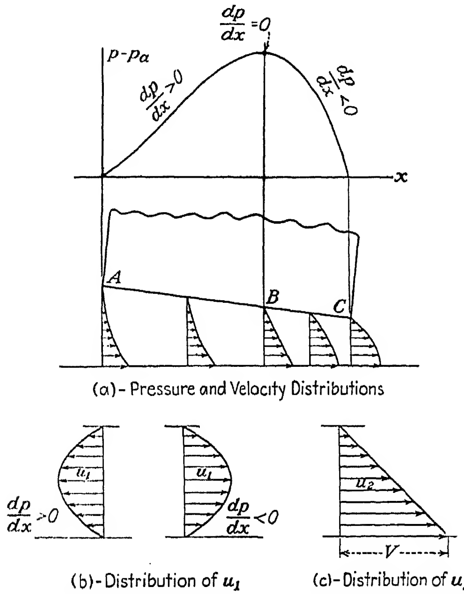


FIG. 263.—Pressure and velocity distribution in an oil film.

dp/dx is zero and the velocity distribution is linear. Curves of these types are shown below the pressure-distribution diagram of Fig. 263*a*. From continuity the area under each of these curves must be the same.

The simplest form of a bearing in which h is a function of x would be one in which the two surfaces are planes but with the bottom of the slipper inclined to the bearing plate, as shown in Fig. 264. The plane of the undersurface of the slipper is assumed to intersect that of the bearing plate at a distance a from the origin and to make an angle δ with that surface. The slipper is moving to the left or, in relative motion, the bearing plate is

traveling to the right. If, as is generally the case, the angle δ is small, then the thickness of the film at any point x is

$$h = (a - x) \tan \delta \cong (a - x)\delta \tag{26}$$

and, when this expression is introduced in Eq. (23), the result is

$$\frac{dp}{dx} = 12\mu\left[\frac{V}{2\delta^2(a - x)^2} - \frac{Q}{\delta^3(a - x)^3}\right]$$

The integration of this equation and the fulfillment of the conditions of atmospheric pressure at the ends of the slipper are

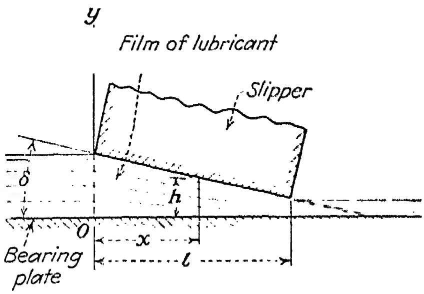


FIG. 264.—The inclined-slipper bearing.

straightforward calculations but, because of their length, the details have been omitted here. The result is obtained by calculating the value of the constant of integration which appears and also by determining the value of Q, the quantity of fluid passing through the interspace. This final result is

$$p = p_a + \frac{6\mu Vx(l - x)}{\delta^2(2a - l)(a - x)^2} \tag{27}$$

Now the mean thickness of the viscous film is found at $x = l/2$ and is equal to

$$\bar{h} = \frac{\delta}{2}(2a - l) \tag{28}$$

If this value is substituted in Eq. (27), then the difference between the pressure at a point in the film and the atmospheric pressure outside it may be expressed as

$$p - p_a = \frac{3\mu Vx(l - x)}{\delta\bar{h}(a - x)^2} \tag{29}$$

It is now proposed to study the effect of variations in the inclination of the slipper for a bearing having constant values of μ, V and $\bar{h}$. For this purpose Eq. (29) may be put in the more convenient nondimensional form

$$\frac{p - p_a}{3\mu\frac{V}{\bar{h}}} = \frac{\frac{x}{l}\frac{l^2}{a^2}\left(1 - \frac{x}{l}\right)}{\delta\left(1 - \frac{l}{a}\frac{x}{l}\right)^2} \tag{30}$$

Before proceeding to the actual computations it is worth while to determine the location of the point of maximum pressure and the value of this pressure. This may be done by differentiating the expression for $p - p_a$, as given by Eq. (29), with respect to x and putting the result equal to zero. The value of x thus obtained is

$$\frac{x_m}{l} = \frac{1}{2 - \frac{l}{a}} \tag{31}$$

When this value is put into Eq. (30) the result obtained is

$$\frac{(p - p_a)_m}{3\mu\frac{V}{\bar{h}}} = \frac{\frac{l^2}{a^2}}{4\delta\left(1 - \frac{l}{a}\right)} \tag{32}$$

In making numerical calculations it is convenient to assume a value of the mean film thickness $\bar{h}$ and the slipper length l.

As an example, consider a case where $\bar{h} = 0.0001$ ft. and $l = 1.0$ ft. The inclination δ of the slipper can then be determined from Eq. (28), after which it is a simple matter to compute the values of $\frac{x_m}{l}$ and $\frac{(p - p_a)_m}{3\mu V/\bar{h}}$ from Eqs. (31) and (32). The results of such calculations, giving the variation of these last two quantities as well as δ with l/a, are shown in Fig. 265. It appears that, as l/a approaches unity, the point of maximum pressure moves toward the rear or right-hand edge of the slipper while at the same time the maximum pressure tends to become infinite. It can be seen from Fig. 264 that the condition $l/a = 1$ corresponds to the case where the slide block is inclined

so that its rear edge just touches the bearing plate. In such a case there could, of course, be no flow through the interspace, so that this limiting case is of no physical significance. The

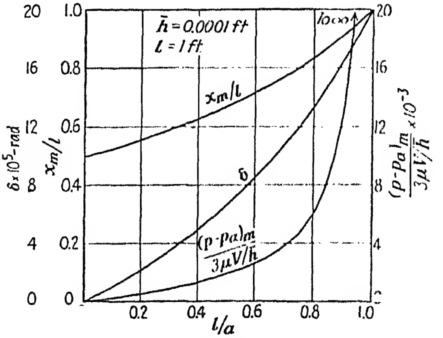


FIG. 265.—Variation of maximum pressure, its location and slipper inclination with l/a.

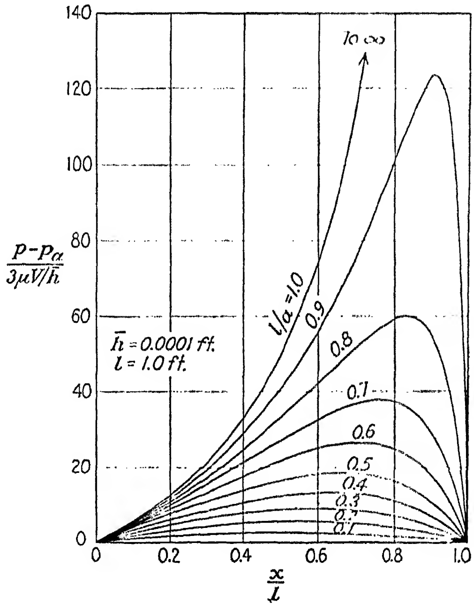


FIG. 266.—Variation of $\dfrac{(p - p_a)}{3\mu V/\bar{h}}$ with $\dfrac{x}{l}$ and $\dfrac{l}{a}$.

other extreme case where $l/a = 0$ corresponds to the parallel slipper and plate and the results given here agree with those previously obtained.

The variation of pressure along the length of the slide block for different inclinations or values of l/a is indicated by the results shown in Fig. 266, these curves having been obtained by introducing into Eq. (30) the numerical values for $\bar{h}$ and l shown in Fig. 265.

In order to complete this study of the bearing, it is of interest to determine the value of the resultant of the pressures acting on the block. This may be done by setting up the integral

$$P = \int_0^l (p - p_a)dx$$

and the result after integrating and simplifying may be put in the form

$$\frac{P}{3\mu\frac{V}{\bar{h}}} \quad \left(2 - \frac{l}{a}\right)\log_e\left(\frac{1}{1 - \frac{l}{a}}\right) - 2\frac{l}{a} \tag{33}$$

The position of the center of pressure or point of application of this resultant force may be found by computing the moment of the distributed load over the slipper about some convenient point and dividing by the resultant force. If the point $x = 0$ is selected for the reference point, then

$$\frac{1}{P}\int_0^l (p - p_a)x\,dx$$

and, after carrying out the integration and introducing the value of P from Eq. (33), it appears that[1]

$$\frac{\left(3 - 2\frac{l}{a}\right. \quad \log_e \quad \frac{1}{1 -} \qquad \frac{l}{2a} \quad \left. - \frac{l}{a}\right)}{\left(2 - \frac{l}{a}\right)\log_e\left(\frac{1}{1 - \frac{l}{a}}\right) - 2\frac{l}{a}} \tag{34}$$

The values of these quantities, $\dfrac{P}{3\mu V/\bar{h}}$ and $\dfrac{x_p}{l}$, for the particular

[1] In making numerical calculations with Eqs. (33) and (34), it is essential that at least seven-place logarithms be employed because of the fact that in each case the two groups of terms forming these expressions differ by very small amounts, particularly for small values of l/a.

case previously considered are plotted in Fig. 267 as functions of l/a. If desired, these values could, of course, be plotted directly as functions of δ.

As a final step in this study the tangential force or resultant of the shearing stresses on the slipper may be evaluated. The shearing stress at any point in the film is

$$\tau = \mu\frac{\partial u}{\partial y} = \left(y - \frac{h}{2}\right)\frac{dp}{dx} - \frac{V}{h}$$

the value of $\partial u/\partial y$ having been computed from Eq. (21). The

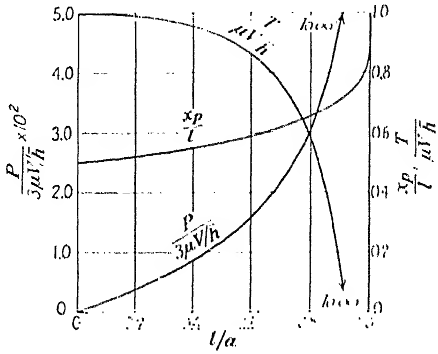


FIG. 267.—Variation of $\frac{P}{3\mu V/\bar{h}}$, $\frac{x_p}{l}$, and $\frac{T}{\mu V/\bar{h}}$ with $\frac{l}{a}$.

value of τ acting on the fluid at a point on the slipper is found by putting $y = h$ so that

$$\tau_s = \mu\left(\frac{h}{2\mu}\frac{dp}{dx} - \frac{V}{h}\right)$$

The total tractive force acting on the slipper is opposite in direction to that on the fluid and is equal to

$$T = -\int_0^l \tau_s\,dx = -\mu\int_0^l\left(\frac{h}{2\mu}\frac{dp}{dx} - \frac{V}{h}\right)dx$$

In order to evaluate this integral it is necessary to introduce the value of dp/dx obtained by differentiating Eq. (27). The final result after simplification is

$$\frac{T}{\mu\frac{V}{h}} = a\quad\frac{l}{a} - \left(2 - \frac{l}{a}\right)\log_e\left(\frac{1}{1 - \frac{l}{a}}\right) \tag{35}$$

The value of $\dfrac{T}{\mu V/h}$ considered as a function of l/a is also shown in the graph of Fig. 267. This force appears to have positive values with the exception of a small range near $l/a = 1.0$. As l/a approaches unity the velocities in the film near the right-hand end of the slipper tend to become infinite. This would require large accelerations and inertia could no longer be neglected.

Normally the tractive force would act from left to right so that, if the slipper were moving over the stationary plate, there would be a force in the opposite direction equal to T which would be necessary to maintain the motion of the slipper. The normal force P also has a component equal to $P\delta$ which opposes the motion so that the net force necessary to move the slipper is

$$H = T + P\delta \tag{36}$$

The theory of the inclined-slipper bearing as presented here is by no means complete. In practice the situation usually is that the total load to be supported by the bearing, the viscosity of the lubricant and the velocity of the slipper may be known, and it is then a question of determining the best proportions for the interspace in which the flow of the lubricant takes place.

219. Practical Aspects of Lubrication.—The theory of lubrication as applied to the problem of Art. 218 is essentially that developed by Osborne Reynolds.[1] It has been elaborated on by many other workers in this field, particularly Sommerfeld, Michell and Kingsbury.[2] Space is not available for more detailed discussions of this work, and the mathematics of such problems as the cylindrical journal bearing and shaft are too advanced for consideration here. The analysis of the inclined-slipper bearing

[1] REYNOLDS, O., On the Theory of Lubrication, *Scientific Papers*, vol. 2, pp. 228–310, Cambridge University Press, 1901.

[2] For discussions of some of this work and more complete bibliographies, the reader is referred to the following:

STANTON, T. E., "Friction," Chaps. III and IV. Longmans, Green & Company, London, 1923.

MICHELL, A. G. M., "Viscosity and Lubrication," Chap. III of "The Mechanical Properties of Fluids," Blackie & Son, Ltd., London, 1923.

KAUFMANN, W., "Angewandte Hydromechanik," vol. II, pp. 177–193, Julius Springer, Berlin, 1934.

HERSEY, M. D., "Theory of Lubrication," John Wiley & Sons, Inc., 1936.

has been idealized in a number of respects. In practice the slipper would be of finite length so that there would be some leakage of the lubricant along the sides of the bearing with the result that the pressures in the film would be considerably diminished. The case of a slide block of finite dimensions has been treated by Michell and details will be found in the references given here.

Another factor which appreciably modifies the theoretical results for bearing performance is that no account has been given of the transformation of the kinetic energy of the fluid into heat. Some of this heat is absorbed by the lubricant with a resulting rise in temperature and a decrease in viscosity. This phase of the problem is closely related to questions of the transfer of heat between solid surfaces and fluids. In many cases the determining factor in bearing design is the necessity for supplying a sufficient amount of lubricant to serve as a cooling agent and to carry away the heat generated rather than to act merely as a means of sustaining the applied load. If, because of excessively high pressures or breaking down of the lubricant at high temperatures, there should cease to be a film between the solid surfaces of the bearing, there is always the danger of seizure and severe damage to the bearing. When metal surfaces are in contact, the problem belongs in the field of solid friction and not in the mechanics of fluids.

The explanation of the ability of a bearing to withstand large loads on the basis of the theory of the flow of viscous fluids is only one phase of the general problem of lubrication. This theory, for instance, does not explain why in many cases bearing troubles may be solved by replacing a mineral oil by a vegetable oil, even though the two fluids have approximately the same viscosity. In order to answer this question it is necessary to enter into the field of physical chemistry and to consider, among other things, the adhesive quality of the oils when in contact with metal surfaces. The explanation of the different characteristics of various oils in this respect involves a consideration of the behavior of the surface molecules of both the oil and the metal.

While the discussion of lubrication which has been presented here covers only a portion of one of the simpler problems in that field, it may serve as an introduction to further study and should

give some idea of the methods that may be used. The value of this theory is well illustrated by the development of the Michell and Kingsbury bearings. When used as thrust bearings in marine-engine installations, these consist of slipper blocks such as that previously discussed except that they are hinged freely about a point some distance back of their centers. Such an arrangement will automatically adjust itself to the load applied and is very stable. It has been said that the application of hydrodynamic theory to bearings of this kind has made it possible to increase bearing pressures from approximately 60 lb. per sq. in. to 3000 lb. per sq. in.

ANSWERS TO PROBLEMS

1. $\rho = 0.00102$ gm. wt. sec.^{2}/cm.^{4}, $w = 1$ gm. wt./cm.^{3}
2. $w = 56.7$ lb. per cu. ft., $\rho = 1.76$ slugs per cu. ft.,
$v = 0.0176$ cu. ft. per lb.
3. $p = 500$ lb. per sq. in., $\rho_1 = 0.422$ slugs per cu. ft.,
$\rho = 0.844$ slugs per cu. ft.
4. $\alpha_C = 0.00366$, $\alpha_F = 0.00218$
5. 17.34 cu. ft.
6. 0.667 ft.^{3}/in.^{2} deg. C..
7. 728 ft. per deg. F.
8. 0.0567 lb. per cu. ft.
9. At 100 lb. per sq. in. abs. (*a*) 2.46 cu. ft. per lb., (*b*) 3.92 cu. ft. per lb.
11. 2740 lb. per sq. ft. abs.
12. 3988 lb. per sq. ft. abs.
13. 14.695 and 14.17
14. 0.866, zero, -1.30, -1.30 lb. per sq. in. gage
15. 33.9 and 67.8 ft.
16. (*a*) 23.1, (*b*) 20.4
17. (*a*) 110.8, (*b*) 10.7
18. 14.4°F., 0.00651°C. per m.
19. 5600 ft.
20. 9.04 lb. per sq. in. abs.
21. 8.28, 1.68, 3.40 lb. per sq. in. abs.
22. 0.0015, 0.000359, 0.00073 slugs per cu. ft.
23. 4.62 ft.
24. 13.55 ft. below *B*
25. 46.2 ft.
26. 3.58 ft., -2.35 ft.
27. 0.849 ft., 1.025 ft. above *B*
28. (*a*) 0.770 ft., (*b*) 0.693 ft.
29. 6.3
30. 0.971 in.
31. 1.42 per cent
32. 30.4 ft.
33. 14.52 lb. per sq. in. abs.
34. (*a*) 3920 lb., (*b*) 1680 lb., (*c*) 3890 lb., (*d*) 4990 lb.
35. (*a*) 6.82 ft., (*b*) 7.52 ft., (*c*) 7.43 ft.
36. 9.00 ft.
38. (*a*) 3.46 ft., (*b*) 7.01 ft. from water surface
39. 11.55 lb.
40. 2090 lb.
41. 196 lb.

42. (*a*) 14,840 lb., (*b*) 15,160 lb., (*c*) 19,180 lb.
43. (*a*) 4.91 lb. per sq. in. gage, (*b*) 4.26 lb. per sq. in. gage, (*c*) 20.2 in., (*d*) 20.58 in.
44. $p_N - p_M = 3.85$ lb. per sq. in.
45. 24 in.
46. 5.19 in.
47. (*a*) 6740 lb., (*b*) 13,480 lb., (*c*) 4.24 ft., (*d*) 4.67 ft.
48. 1181 lb.
49. (*a*) $y_p = 4.50$ ft., $x_p = 1.50$ ft. from vertical edge;
(*b*) $y_p = 7.29$ ft., $x_p = 1.43$ ft. from vertical edge
50. (*a*) $P = 12{,}480$ lb., $y_p = 6.67$ ft. from hinge;
(*b*) $P = 24{,}960$ lb., $y_p = 5.83$ ft. from hinge;
(*c*) 10,400 lb.
51. 12,480 lb.
52. (*a*) 2496 lb., (*b*) 2.80 ft. from top, (*c*) 3.17 ft. from top of box
53. $P = 26{,}960$ lb., $y_p = 7.0$ ft.
54. $y_p = 3.75$ ft., $\bar{y} = 9.0$ ft.
55. 1774 lb.
56. $P_x = 187.2$ lb., $P_y = 196.0$ lb., $P_z = 187.2$ lb.
57. 29.9 ft.
58. 2650 lb. per sq. in. abs.
59. 5 lb. per in.
60. (*a*) Zero, (*b*) $P_x = 17{,}470$ lb., $P_y = 20{,}320$ lb., (*c*) 26,800 lb.
61. 4.38 ft., 4.38 ft.
62. 499 lb.
63. 130.7 lb.
64. 1028 lb.
65. 1045 lb., 30,200 lb.
66. 28.2 lb.
67. 4450 lb.
68. 72.9 cu. ft.
69. $MG = 0.606$ in., $T = 34.2$ in. lb.
70. (*a*) $0.101b$, (*b*) $-0.083b$, (*c*) $0.019b$
71. 4 ft., 11 ft.
72. 3.04 ft.
73. $\frac{4}{5}$
74. (*a*) 7.04 ft., (*b*) 34.4 lb. vertically upward
75. 1.225
76. $d_1 = 1.60$ ft., $d_2 = 90.4$ ft.
77. (*a*) 1.093 ft. from bottom, (*b*) 1.284 ft. from DE,
(*c*) 1.377 ft. from DE
78. 0.917 ft. from DE, 2250 lb.
79. 23 plates
80. 104.6 ft. tons
81. $T = 247$ ft. lb., 17°37′ from vertical
82. 177.6 lb.
83. 1 ft.
84. $A_y = 1140$ lb. down, $A_x = 4520$ lb. to left, $B_x = 5780$ lb. to left

85. 3200 lb.
86. 3.17 ft., 1.19 ft.
87. 13.4 ft.
88. 699 lb.
89. Sp. gr. = 0.567, more stable
90. 43.3 lb. per cu. ft., 89.1 lb.
91. 2570 lb.
92. 12 in., 46.8 lb.
93. 23.4 lb. per sq. ft. gage.
94. 153.1 r.p.m., 249.6 lb. per sq. ft. gage, 108.3 r.p.m., 1 ft.
95. 177.2 r.p.m., 1.57 cu. ft.
96. 13.5 in., $p_B = 6.63$, $p_C = 5.89$, $p_D = 5.15$ lb. per sq. in. gage
97. $p_B = 5.89$, $p_C = -3.26$ lb. per sq. in. gage.
98. 8.02, 14.98 rad. per sec.
99. 5.67 rad. per sec., $p_C = 2.95$, $p_D = 11.8$ lb. per sq. in. gage
100. 8.02 rad. per sec.
101. (*a*) 2.32 rad. per sec., (*b*) 6.75 rad. per sec., (*c*) 8.18 rad. per sec.
102. 8°18′ from horizontal, 230 lb. per sq. ft. gage
103. $p_B = 0.43$, $p_A = 2.16$ lb. per sq. in. gage
104. 11.34 rad. per sec., 1.62 lb. per sq. in. gage
105. 8.02 rad. per sec., 0.325 lb. per sq. in gage at midpoint of *BC*
106. 8.02 rad. per sec.
107. 17.4 ft. per sec.^{2}
108. At inlet $V = 0.164$. At outlet $V = 1.47$ ft. per sec.
109. At base $V = 618$ ft. per min., $Q = 1{,}213{,}000$ cu. ft. per min.
At 50 ft. level, $V = 688$ ft. per min., $Q = 1{,}046{,}000$ cu. ft. per min.
110. 130, 104, 86.6 ft. per sec.
111. 22.13 ft. or 1383 lb. per sq. ft.
112. $H = 10.24$ ft. for all points
113. $E = 2117$ lb. per sq. ft. for both points
114. 14.32 lb. per sq. in.
115. 13.3 lb. per sq. in.
116. 4.45 lb. per sq. ft. gage
117. 24.06 ft. per sec., 4.72 c.f.s., 8.20 lb. per sq. in. abs.
118. 24.06 ft. per sec., 1.18 c.f.s., 11.87 lb. per sq. in. abs.
119. 7.85 c.f.s., p_s = zero abs., $h = 24.91$ ft.
120. 24.06 ft. per sec., 1.18 c.f.s., 12.57 lb. per sq. in. abs.
121. 6.22 ft., 16.22 ft. gage
122. 114.6 ft. per sec.
123. 171 m.p.h.
124. 94.7 ft. per sec.
125. 5 ft. gage, 8.50 ft. gage
126. 185.2 m.p.h.
127. 22.6 lb. per sq. in. gage
128. 3.18 ft. per sec., 5.62 c.f.s.
129. 0.0403 in.
130. 24.2 lb. per sq. in.
131. 4.13 ft.

132. 2.41 ft. per sec.
133. 0.52 lb. per sq. in. abs.
134. (*a*) 12.53 ft. per sec.
135. 18.7, 24.3 ft. per sec.
136. 42.0 lb. per sq. in. gage, loss = 11.8 ft.
137. (*a*) 16.04 ft. per sec.
138. (*a*) 20.14, (*b*) 21.33, (*c*) 18.92 ft. per sec.
139. 9.78 ft.
140. 3.83 ft.
141. (*a*) 6.09 c.f.s., $p_C = 8.20$, $p_D = 6.02$ lb. per sq. in. abs., (*b*) 2.79 ft.
142. 2.36 c.f.s.
143. $C = 1.023$
144. 362 ft. per sec., 450 sec.
145. 183 ft. per sec., 73 ft. per sec.
146. 2.79 lb. per sq. in., 0.693 lb. per sq. in.
147. 8.19, 16.88 lb. per sq. in. abs., $Q = 3.50$ c.f.s.
148. 1.72 c.f.s.
149. 24.1 hp.
150. 26.67 ft. per sec., 133 ft. lb.
151. 65.2 ft. lb., 32.6 lb. sec.
152. 1.55 lb.
153. 4.84 lb.
154. 38.0 lb.
155. 358 lb.
156. $P_x = 38.7$ lb., $P_y = 144.2$ lb.
157. $P_x = 100.0$ lb., $P_y = 111.6$ lb.
158. 134.3 lb.
159. 80 g.p.m., 13.8 lb.
160. 87.2 ft. per sec., 37.2 ft. lb. per lb., 36°35′
161. $P_x = 463$ lb., $P_y = 192$ lb., 42.1 hp.
162. 3690 lb., 1007 hp.
163. $v = V/3$
164. 1130 lb., 2120 lb.
165. 153 lb.
166. $R_x = 2459$ lb., $R_y = 1065$ lb.
167. 155 lb.
168. 93.0 lb.
169. 93 lb., 3.98 ft.
170. $V' = 52.9$ ft. per sec., $P_x = 523$ lb., $P_y = 304$ lb., 111.8 ft. lb. per lb., 38.1 hp.
171. 43.2 lb., 14.9 lb., 0.94 hp.
172. $P = 562$ lb., work per sec. = 12,360 ft. lb., K.E. per sec. = 8150 ft. lb., eff. = 60.3 per cent.
173. 2.20 hp., $V' = 67.4$ ft. per sec.
174. $\alpha = 143°06'$, 16,100 ft. lb., eff. = 55.4 per cent, $R_y = 48.6$ lb.
175. (*a*) 565 lb., 2262 lb., (*b*) 1090 lb., 2509 lb.
176. $\beta_1 = 60°$, $u_1 = v_1 = 17.32$ ft. per sec., $\omega = 11.55$ rad. per sec., $u_2 = 8.66$ ft. per sec., $V_2 = 14.44$ ft. per sec., $p_2 = 14.2$ lb. per sq. in., $T = 3150$ ft. lb., 66.2 hp.

177. (a) $L = 74.7$ lb., $D = 6.54$ lb., (b) $L = 72.4$ lb., $D = 19.2$ lb.
178. $\theta = \pm 30°$ and $\pm 150°$, $(p - p_0)_{\text{max.}} = \rho V_0^2/2$ at $\theta = 0$ and $180°$, $(p - p_0)_{\text{min.}} = -3\rho V_0^2/2$ at $\theta = \pm 90°$
179. 60 ft. per sec.
180. $\theta = \sin^{-1}(-\Gamma/4\pi a V_0)$, $V_p/V_0 = 2$
181. 0.465 and 0.055 lb. per sq. in.
182. $L = 6.90$ lb., $C_L = 2.16$
183. -5.25 when $\theta = 30°$
184. $C_L = 2.06$, 251 r.p.m.
185. 0.559
186. $L = 16.1$ lb., $C_L = 0.986$
187. $L = 107$ lb., $C_L = 2.22$, $\alpha_0 = 20°43'$
188. $C_L = 0.548$ when α_0 is $5°$
189. -0.97 deg.
190. 2160 lb.
191. 0.829
192. (a) $5°35'$, 621 ft. per sec., (b) $T = 75.2$ lb., $F_Q = 33.9$ lb.
193. $\alpha = 5°33'$, $V_0/nD = 0.431$
194. $\theta = 15°59'$ at $0.75r$
195. 2.73 ft. per sec., 30.70 ft. per sec.
196. $C_T = 0.155$, $\eta = 0.87$, $V_0/nD = 0.759$, $E = 404$ hp.
197. $L = 400$ lb., 82.8 lb.
198. $\Gamma = 16.2$ ft.^{2} per sec., $L = 0.771$ lb.
199. (a) 1.39, (b) 225 sq. ft.
200. 16,700 lb.
201. 3.13 lb.
202. $T = 25.0$ lb., $F_Q = 9.51$ lb., $\alpha = 2°$
203. $\alpha = 10°52'$ at 2000 r.p.m., max. $N = 2180$ r.p.m.
204. $C_T = 0.329$, $C_E = 0.413$, ideal $\eta = 83.5$ per cent, actual $\eta = 75$ per cent
205. 7.68 lb. per sq. ft.
206. $\nu = 1.11$ stokes, $\varphi = 1.12$ cm. sec. per g.
207. $\nu = 0.01147$ stokes or 1.232×10^{-5} ft.^{2} per sec., $\mu = 2.388 \times 10^{-5}$ slugs per ft. sec.
208. $\mu = 0.028$ slugs per ft. sec. or 13.43 poises, $\nu = 15.8$ stokes
209. 461 lb. per sq. ft., 1.5 ft. per sec.
210. At 3 cm. from axis, $u = 64$ cm. per sec., $\tau = 60$ dynes per sq. cm.
211. 0.197 ft. per sec., 5.06 ft. per sec.
212. 1.08 ft. per sec., 0.177 ft. per sec.
213. 1,400,000
214. 39,200
215. 0.00385 ft. per sec., 0.0030, 1.16×10^6 ft.
216. $\tau = 0.1038$ lb. per sq. ft., 0.0543 in. lb.
217. 0.158 poises.
218. (a) 0.988 ft. per sec., (b) 1.6 ft. per sec.
219. 18.9 ft.
220. 764 lb. per sq. in.
221. 30.7 lb. per sq. in., 61 lb. per sq. in.
222. 5.02 c.f.s.

223. (*a*) 26 ft., (*b*) 22 ft.
224. (*a*) 317 ft., (*b*) 75.1 ft.
225. 3.9 c.f.s.
226. 2.24, 2.47 ft. per sec.
227. 0.50 ft. drop, 0.80 ft. rise
228. 0.40 ft. drop, 1.69 ft. drop
229. 6.29 c.f.s., 1.05 ft.
230. $H = 12.2$ ft.
231. (*a*) 4.2 c.f.s., (*b*) 4.3 c.f.s.
233. $Q = 8.25$ c.f.s., $p_C = 11.7$ lb. per sq. in. abs
234. 0.57 ft.
235. 50 ft.
236. 11.5 c.f.s.
237. 195 ft.
238. 0.016.
239. 11.3 lb. per sq. in., 0.00402 ft. lb. per lb.
240. 67 lb. per sq. in. gage
241. (*a*) 1.18 ft., (*b*) 0.76 ft.
242. 0.048 ft.
243. 9.75 c.f.s.
244. 63.3 ft.
245. (*a*) 1.20 c.f.s., (*b*) 0.42 c.f.s.
246. 0.0273 ft.
247. 0.0101 c.f.s.
248. 110 ft.
249. 7.9 c.f.s.
250. 6.5 c.f.s.
251. Old, $Q = 6.56$ c.f.s.
252. 173 ft.
253. 1.19 ft.
254. 0.000164.
255. 0.000448.
256. $Q = 108$ c.f.s., $\tau = 0.20$ lb. per sq. ft.
257. 0.000606.
258. 94.3 c.f.s.
259. 4.39 ft., more
260. $Q = 907$ c.f.s., $s = 0.00208$
261. 0.775 ft., 4.383 ft.
262. 4.36 ft.
263. $dy/dx = 0.00896$, deeper
264. 98.5 c.f.s., 98.2 c.f.s., 95.8 c.f.s.
265. 45.85 c.f.s.
266. 2.73 ft.
267. 4.43 ft.
268. 2.40 ft., 3.72 ft.
269. 1.10 ft.
270. $C = 3.60$, $H = 3.36$ ft.

271. (a) 68.8 c.f.s., (b) 59.3 c.f.s.
272. 1.05 ft. below channel
273. 3.29 ft. below pond surface
274. 0.000608, 0.000467
275. 257°28′, 0.813d.
276. Flume 0.46 ft. below channel
277. y_c = 1.55 ft., V_c = 4.99 ft. per sec.
278. d_2 = 2.53 ft., s = 0.01123
279. d_1 = 0.498 ft., h_l = 5.37 ft.
280. 4.19 ft.
281. 73 weirs, H = 0.482 ft.
282. L = 14.4 ft., H_V = 1.82 ft., H_R = 0.302 ft.
283. Q = 493 c.f.s., s = 0.00175
284. 34.7 sq. ft., 36.6 sq. ft.
285. 2.90 ft.
286. 7.21 ft. at 200, 6.41 ft. at 400
287. d_c = 3.11 ft., q = 33.6 c.f.s. per ft.
288. s_c = 0.00301, d_2 = 0.883 ft., E_1 = 1.25 ft., E_2 = 0.96 ft., h_l = 0.29 ft.
289. $d = b$, $d = 1.26b$
290. (a) 38.0 ft. per sec., (b) 43.3 ft. per sec.
291. V = 19.6 ft. per sec., h_l = 12.9 ft.
292. d = 1.57 in., V = 25.0 ft. per sec., Q = 0.336 c.f.s.
293. C_v = 0.973, C_c = 0.617
294. V_1 = 2.55 ft. per sec., V = 16.0 ft. per sec., H = 4.14 ft.
295. 4.79 c.f.s., 5.08 c.f.s.
296. 8.87 lb. per sq. in.
297. 1.10 ft., 0.937.
298. 16.0 ft. per sec., 0.234 ft., 0.973.
299. (a) Q = 0.573 c.f.s., h_l = 5.2 ft., (b) Q = 0.424 c.f.s., h_l = 0.5 ft., $Q_{max.}$ = 0.924 c.f.s.
300. 11.15 lb. per sq. in. abs.
301. (a) 1.40 c.f.s., (b) 15.7 ft., (c) 10.9 ft.
302. Q_1 = 0.840 c.f.s., h_l = 2.56 ft.; Q_2 = 0.764 c.f.s., h_l = 0.236 ft.
303. 230 sec.
304. 728 sec.
305. 0.969, 0.650
306. 0.625
307. 1.58 in., 6.26 lb. per sq. in., 0.57 ft.
308. 985 sec.
309. (a) 0.872 c.f.s., (b) 0.721 c.f.s.
310. $d_{max.}$ = 7.86 ft., $d_{min.}$ = 6.48 ft., $Q_{max.}$ = 2.28 c.f.s., $Q_{min.}$ = 2 c.f.s.
311. 9.79 c.f.s., 245 hp.
312. Loss in pipe = 4.87 ft., loss in nozzle = 17.5 ft., hp. delivered = 207, hp. lost = 7.43 per cent
313. (a) 4.00 ft., 0.248 c.f.s., (b) 3.58 ft., 0.162 c.f.s., (c) 4.50 ft., 0.304 c.f.s., (d) 4.08 ft., 0.215 c.f.s.
314. 276 sec.
315. 11.5 lb. per sq. in. gage

316. 132 sec.
317. 22.6 ft. per sec., 0.304 c.f.s.
318. (*a*) 0.501 c.f.s., (*b*) 0.501 c.f.s.
319. 1.138 in., 18.9 lb. per sq. in.
320. (*a*) 125 c.f.s., (*b*) 177 c.f.s., (*c*) 161 c.f.s.
321. (*a*) 9380 lb., (*b*) 11.5 lb.
322. $k_D = 0.54$, $C_D = 1.08$
323. (*a*) 35.45 m.p.h., (*b*) 35.30 m.p.h.
324. 149 lb. at 60 m p.h.
325. 3,520,000 for air, 44,600,000 for water
326. $C_D = 64$, $D = 0.0000224$ lb.
327. 12.3 ft. per sec.
328. At $y = 1.0$ in., $u_l = 7.50$, $u_t = 9.05$ ft. per sec.
329. At $y = 0.50\delta$, $u_l = 18.7$, $u_t = 22.6$ ft. per sec.
330. At $y = 1.5$ in., $\tau = 0.001375$ lb. per sq. ft.
331. At $\theta = 50°$, $p = -4.00$ lb. per sq. ft., $V = 76.6$ ft. per sec., K.E. = 6.97 ft. lb. per cu. ft., total energy = 2119.4 ft. lb. per cu. ft.
332. At $x = 27$ in., $\delta_l = 0.252$ in., $\delta_t = 0.792$ in.
333. At center, $\tau_l = 0.0379$ lb. per sq. ft., $\tau_t = 0.213$ lb. per sq. ft., $C_{f_l} = 0.000789$, $C_{f_t} = 0.00379$
334. For laminar flow, $C_f = 0.00265$, (*a*) $C_f = 0.00188$, (*b*) $C_f = 0.00188$, (*c*) $C_f = 0.00375$. For turbulent flow, C_f 0.0061 (*a*) $C_f = 0.00536$, (*b*) $C_f = 0.00536$, (*c*) $C_f = 0.00708$
335. (*a*) 0.00495, (*b*) 0.00268
336. (*b*) 0.00297 for $N_C = 500{,}000$, 0.00379 for $N_C = 250{,}000$
337. (*a*) $\delta = 0.0866$ ft. at $x/l = 0.25$
$\delta = 0.1500$ ft. at $x/l = 0.75$
(*b*) $\delta = 0.0274$ ft. at $x/l = 0.25$
$\delta = 0.1890$ ft. at $x/l = 0.75$
(*c*) $\delta = 0.0494$ ft. at $x/l = 0.25$
$\delta = 0.1189$ ft. at $x/l = 0.75$
338. For $N_R = 10^7$, $C_f = 0.00294$ by $\frac{1}{7}$-power law, $C_f = 0.00300$ by Schlicting's formula, 2 per cent at $N_R = 10^7$, 25.5 per cent at $N_R = 10^9$
339. (*a*) 575 lb., (*b*) 567 lb.
340. 27.6 ft. per sec.
341. 22.4 ft.
342. $D = 13.3$ lb. per ft. at 5 ft. per sec., $D = 62.2$ lb. per ft. at 20 ft. per sec.
343. $D = 54.0$ lb., $V = 564$ ft. per sec.
344. 0.408 lb. for sphere, 0.106 lb. for ellipsoid
345. 214,700 lb.
346. $V = 7.84$ ft. per sec., $C_D = 60$, $D = 0.0000154$ lb.
347. 4890 lb., 1170 hp.
348. $L = 175$ lb. per ft., $D = 9.59$ lb. per ft., $L_{\text{max.}} = 389$ lb. per ft.
349. $D_i = 244$ lb., $C_{D_i} = 0.0254$
350. For $R = 8$, $C_{D_p} = 0.0718$, $C_{D_i} = 0.0407$, $C_D = 0.1125$
351. (*a*) 0.359, (*b*) 4.81 m.p.h.
352. $D_f = 3070$ lb., $D_r = 2930$ lb., $C_f = 0.00182$, $C_r = 0.00174$
353. (*a*) $D = 281$ lb., $C_D = 0.535$, (*b*) 43.2 hp. at 80 m.p.h.

354. 0.0062.
355. $D_A/D_1 = 0.38$
356. (*a*) 515 lb., (*b*) 0.0313
357. $D_p = 220$ lb., $D_i = 75$ lb., $D = 295$ lb.
358. $\Delta v = 0.0000467$ cu. ft., $p = 1067$ lb. per sq. in. abs.
359. $E_1 = 6480$ lb. per sq. ft., $E_2 = 2880$ lb. per sq. ft., $v_1 = 4.77$ cu. ft. per lb., $v_2 = 10.73$ cu. ft. per lb.
360. $E_1 = 9110$ lb. per sq. ft., $E_2 = 2812$ lb. per sq. ft., $v_1 = 4.77$ cu. ft. per lb., $v_2 = 8.48$ cu. ft. per lb.
361. For isothermal air, $E = 2116.4$ lb. per sq. ft.
For adiabatic air, $E = 2975$ lb. per sq. ft.
362. 4720, 1455 ft. per sec.
363. At 100°F., $c = 1165$ ft. per sec.
364. 1.252
365. 797, 796 ft. per sec.
366. 12.9 lb. per sq. in. abs.
367. $N_{M_A} = 0.286$, $N_{M_B} = 0.410$
368. 23.2 lb. per sq. in. abs.
369. 131 ft. per sec., 18.6 lb. per sec.
370. (*a*) Incompressible, $V = 133.8$ ft. per sec.; compressible, $V = 127.7$ ft. per sec. (*b*) Incompressible, $V = 216.0$ ft. per sec.; compressible, $V = 194$ ft. per sec.
371. $p_1v_1 \log_e (p_1/p_2)$
372. $(p_1v_1 - p_2v_2)/(k - 1)$
373. 50 lb. per sq. in. abs., 71,690 ft. lb.
374. 37.8 lb. per sq. in. abs., 62,450 ft. lb.
375. 264.4 lb. per sq. in. abs., 0.01226 slugs per cu. ft.
376. 28.3 B.t.u.
377. $I_2 - I_1 = 11.8$ B.t.u. per lb., external work $= 10{,}260$ ft. lb. per lb.
378. $I_1 - I_2 = 67.8$ B.t.u., external work $= 52{,}750$ ft. lb.
379. For 500 ft. to 750 ft., (*a*) $\Delta p = 7.7$ lb. per sq. in.,
(*b*) $\Delta p = 9.3$ lb. per sq. in.
381. 1795 ft.
382. $l = 33.6$ ft., $V_1 = 15.24$ ft. per sec., $V_2 = 15.26$ ft. per sec.
383. (*a*) $p_0 = 6.94$ lb. per sq. in. abs., $V_0 = c_0 = 1416$ ft. per sec.,
(*b*) $l = 3720$ ft., (*c*) At $V/V_1 = 5$, $l = 3620$ ft., $p = 23.63$ lb. per sq. in. abs., $V = 475$ ft. per sec., $c = 1513$ ft. per sec., $v = 15.80$ cu. ft per lb.
384. 1890
385. $D \propto V^{1.80}$, $n > 2$ in transition range
386. $\dfrac{D}{\rho V^2 b^2}$, $\dfrac{\rho V b}{\mu}$, $\dfrac{h}{b}$ where h = height, b = width
387. $D = K\rho V^2 a^2 \left(\dfrac{b}{a}\right)^h \left(\dfrac{d}{a}\right)^i \div \left(\dfrac{V}{\sqrt{ag}}\right)^k$
388. (*a*) 0.177, (*b*) 0.5, no
389. 5.94 c.f.s.
390. (*a*) $s' = s\lambda^{1\frac{2}{3}}$, (*b*) $s' = s\lambda^{\frac{1}{3}}$

393. $h = \dfrac{K}{N_R{}^f}\left(\dfrac{l}{d}\right)^h\left(\dfrac{\epsilon}{d}\right)^j\dfrac{V^2}{2a.}$

394. 5.98 m.p.h

395. 17.4 ft. per sec 2770 lb.

396. $Q = K\sqrt{2gH}\ H^2\ N_R{}^{-c}N_F{}^{-2(c+e+f)}\left(\dfrac{H}{h}\right)^f$

397. $V' = 3000$ m.p.h., $N_M = 0.328$, $N_M{}' = 3.93$

398. $V' = 250$ m.p.h., $p' = 12p$

399. At 60°F., $V' = 850$ m.p.h.; at 120°F., $V' = 429$ m.p.h.

400. $T = K\rho V^2 d^2\left(\dfrac{V}{nd}\right)^{2-e-f} N_R{}^{-f}N_M{}^{-i}$

401. $Q = K\omega H^3\left(\dfrac{H}{D}\right)^{-c-e+f} N_R{}^{-e}N_F{}^{-2f}$

402. 177 sec.

403. 1.20 ft.

404. 2.911×10^{-9} slugs per ft. sec.

405. (*a*) $A = 0.00513$, $B = 1.99$, (*b*) $A = 0.000615$, $B = 0.239$

406. 0.384 stokes

407. $A = 5.25$, $B = 917$

408. 330 sec.

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